A NUMERICAL STUDY OF BUNCHED BEAM TRANSVERSE *e-p* INSTABILITY BASED ON THE CENTROID MODEL *

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Abstract

In a recent theoretical study of the transverse electron-proton (e-p) instability, an asymptotic solution has been found for the equations describing the centroid motion of the traversing proton bunch and the stationary background electrons.[1] It was shown that the combination of finite proton bunch length, non-uniform proton line density, and the single-pass e-p interaction cause the instability to evolve intricately in space and time even in the linear regime. This paper reports a numerical study of the e-p instability based on the same centroid equations. The purpose of the work is to compare the numerical solution with the analytic solution and to use the numerical approach to investigate the early development of the instability not covered by the asymptotic solution. In particular, the instability threshold and the initial growth of the instability are studied for various proton-beam conditions, fraction of charge neutralization, and initial perturbations.

INTRODUCTION

In recent years, there has been a growing interest in studying the transverse electron-proton (e-p) two-stream instability in intense proton beams. One of the focuses is on the e-p instability observed in the long proton bunch like the one in the Proton Storage Ring (PSR) at Los Alamos National Laboratory [2]. Although the basic mechanism of the instability has been well known, the theory for a bunched beam e-p instability is still under developing. In a recent theoretical study of transverse e-p instability, an asymptotic solution has been found for the equations describing the centroid motion of the traversing proton bunch and the stationary background electrons.[1] The growth rate and the stability threshold were estimated based on the asymptotic solution. As discussed in Ref. 1, the results derived from this kind of approach are applicable to special cases only. The initial evolution of the perturbations and the instability threshold still need more investigation. The work reported in this paper is a numerical study based on the centroid equations discussed in Ref. 1. We will compare the numerical results with the analytic solution and study the early development of perturbations in the proton bunch by investigating various proton-beam conditions, fraction of charge neutralization, and initial conditions. Since the numerical approach inevitably lacks generality, the intention here is to extract some qualitative understanding in a limited parameter space only.

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CENTROID MODEL

We consider a bunched proton beam of full length L and circular cross section of radius a, propagating with a constant velocity **v** through a stationary electron background of infinite extent in the direction of beam propagation. The protons are confined in the transverse direction by a linear external focusing force. We assume that in the equilibrium state, particles are distributed uniformly in the transverse direction and the electrons experience a linear transverse focusing force due to the space charge of the proton bunch. A Cartesian coordinate system is chosen such that the z axis is pointing opposite to the direction of proton propagation, and the origin coincides with the center of the beam cross section. The line densities of the protons and electrons, λ_p and λ_e, generally depend on z. The synchrotron motion of the protons and the axial motion of the electrons in the laboratory frame are neglected for simplicity. We also neglect the impedance due to the beam environment, and consider the transverse motion in only one direction, say the y direction. The stability study is based on a model in which each electron interacts with the proton beam only once, i.e., a "one-pass" interaction between the electrons and protons.

The centroid of the proton beam $Y_p(z,t)$ and the centroid of electrons $Y_e(z,t)$ are defined by

$$Y_q(z,t) = \int_{-\infty}^{\infty} y_q(z,t,\omega_q) F_q(\omega_q) d(\omega_q/\Delta_q) \ , \qquad (1)$$

where the subscripts q stands for p (protons) or e (electrons), $y_q(z,t,\omega_q)$ is the particle displacement at the position z and time t, ω_q is the oscillation frequency, $F_q(\omega_q)$ is the frequency distribution function, and Δ_q characterizes the frequency spread of ω_q. We consider a Lorentzian distribution function $F_q(\omega_q) = (\Delta_q^2/\pi)[\Delta_q^2 + (\omega_q - \omega_{qo})^2]^{-1}$, where ω_{qo} is the mean value of ω_q. Averaging over the equations of single particle motion yields

$$D^2 Y_p + 2\Delta_p D Y_p + (\omega_\beta^2 + \Delta_p^2) Y_p = \omega_\beta^2 \xi(z) Y_e \ , \qquad (2)$$

and

$$\ddot{Y}_e + 2\Delta_e \dot{Y}_e + [\Omega^2(z) + \Delta_e^2] Y_e = \Omega^2(z) Y_p \ , \qquad (3)$$

where $D = \partial/\partial t - v(\partial/\partial z)$, and ω_β is the undepressed betatron frequency, $\xi(z) = 2r_p c^2 \lambda_e(z)/(a^2\omega_\beta^2\gamma)$, $\Omega(z) = (c/a)\sqrt{2r_e\lambda_p(z)}$, is the electron bounce frequency, c is the speed of light, $\gamma = (1 - v^2/c^2)^{-1/2}$; r_p and r_e are the classical radii of a proton and an electron, respectively. In deriving Eqs. (2) and (3), we have also assumed that the incoherent betatron frequency shift due to the self-fields of the proton beam is negligible, and that the maximal value of λ_e is much smaller than that of λ_p, so that $\omega_{po} = \omega_\beta$. The perturbing forces are assumed to be $m_e\Omega^2(z)Y_p$ for electrons and $m_p\omega_\beta^2\xi(z)Y_e$ for protons, where m_q is the relativistic mass of a proton or an electron. Note that Eqs. (2) and (3) depend on the choice of the frequency distribution functions.

An approximate asymptotic solution for Eqs. (2) and (3) in the beam frame is found to be

$$Y_p(z',t) \approx C_p M_p(z')\, \mathrm{e}^{-\Delta_p t}\left\{\left[\frac{I_1(u)}{u} - \frac{\mathcal{J}^2 I_2(u)}{8u^2}\right] \times \cos T_p - \left[\frac{J_1(u)}{u} - \frac{\mathcal{J}^2 J_2(u)}{8u^2}\right]\cos S_p\right\}, \quad (4)$$

where z' is the distance from the head of the proton bunch, $J_n(x)$ and $I_n(x)$ are Bessel functions, $u = \sqrt{2\theta\mathcal{J}}$, $\theta = \theta(z',t) = \omega_\beta(t - z'/v)$, $M_p(z') = \mathcal{J}\xi(z')R(z')\exp\big[(\Delta_p - \Delta_e)z'/v\big]$, $T_p = P - \theta$, $S_p = P + \theta$, $P = \sigma_p + \Theta(z') - \mathcal{J}/4$, C_p and σ_p are constants, $R(z')$ and $\Theta(z')$ are determined by $\Phi(x) = R(x)\mathrm{e}^{i\Theta(x)}$ and $\Psi(x) = R(x)\mathrm{e}^{-i\Theta(x)}$,

$$\mathcal{J} = \mathcal{J}(z') = i\int_0^{z'/v} \frac{\Omega^2(x)\xi(vx)}{W(x)}[R(x)]^2 dx\ , \quad (5)$$

$i = \sqrt{-1}$, $\Phi(x)$ and $\Psi(x)$ are the linearly independent solutions of the equation $d^2Y/dx^2 + \Omega^2(x)Y = 0$, and $W(x)$ is the Wronskian of $\Phi(x)$ and $\Psi(x)$. In deriving Eq. (4), we have assumed that $Y_e = dY_e/dt = 0$ for $z' \le 0$. The solution for Y_e is very similar to Eq. (4). The growth (or damping) rate $\Gamma_p(z',t)$, and the stability threshold of the proton motion $(\Delta_p)_t$, can be estimated from Eq. (4) as

$$\Gamma_p(z',t) \approx -\Delta_p + \frac{\omega_\beta\mathcal{J}\big[8uI_2(u) - \mathcal{J}^2 I_3(u)\big]}{u\big[8uI_1(u) - \mathcal{J}^2 I_2(u)\big]}\ , \quad (6)$$

and

$$(\Delta_p)_t \approx \mathrm{Max}\left[\frac{\omega_\beta\mathcal{J}}{4}\left(\frac{1 - \mathcal{J}^2/48}{1 - \mathcal{J}^2/32}\right)\right], \quad (7)$$

where $[Y_p(z',t)]_a$ denotes the amplitude of $Y_p(z',t)$, and $\mathrm{Max}[f(z')]$ indicates the maximum of $f(z')$. For $u \gg 1$, we have $\Gamma_q(z',t) \approx \omega_\beta\sqrt{\mathcal{J}/(2\theta)} - \Delta_p$. Typically, $\mathcal{J} < 1$ for a small fractional charge neutralization, $\Gamma_p(z',t)$ is a monotonically decreasing function of time, and the highest growth rate occurs at the tail of the proton bunch. Eqs. (6) and (7) are valid only when the centroid motion can be described by the asymptotic solution given in Eq. (4).

NUMERICAL STUDY

The study here is based on the numerically solution of Eqs. (2) and (3). Earlier numerical studies as well as simulations using the similar equations, including the variations that cover multi-electrons and electron production, have yielded reasonable agreement with experimental data.[3,4]

Comparison between numerical results at large time and the asymptotic solution given in Eq. (4) shows good qualitative agreements: the growth rate depends linearly on Δ_p as a consequence of choosing the Lorentzian frequency spreads, the perturbation wavelength along the proton bunch is roughly proportional to the square root of the proton line density, and the damping of instability in the long time as described in the analytic solution. At large t, the numerical solutions show that the growth or the damping rate depends weakly on Δ_e.

Our main interest is to investigate the initial evolution of perturbations. The following is a summary of general characteristics observed in the numerical solutions:

(i) The initial growth or damping rate does depend on Δ_e. The dependence diminishes as time increases. This result is not covered by the asymptotic solution in Eq. (4).

(ii) The dependence of the growth rate on Δ_p is not affected by the initial conditions.

(iii) Since the growth rate given in Eq. (6) is a local quantity, and because we are considering the one-pass interaction between the electrons and the protons, initial perturbations with quarter wavelength comparable to the bunch length have strong influence on the initial growth rate. Further, the growth rate at any location is not affected by the perturbations behind. Typically, constant-amplitude (or constant-envelope) sinusoidal perturbations with wavelength substantially smaller than the bunch length are initially damped at a rate near Δ_p. For similar perturbations having tilted envelope, the initial damping rate is shifted from Δ_p by certain amounts with a sign opposite to that of envelope's slope. The initial slope of the damping rate also shows the same kind of dependence on the envelope of initial perturbations.

(iv) In the beginning, there is a transient period before the centroid motion, and hence the growth rate, is evolved into the asymptotic regime. During this transient period, the evolution of the growth rate depends on the density profile and other parameters in a complicated way, e.g. the growth rate may oscillate. The system stability can not be judged base on the momentary sign of the growth rate.

(v) The length of the transient period for the growth rate to evolve into the asymptotic regime appears to be independent of Δ_p. Variations of other parameter values, like the fraction of neutralization, that make the system less stable tend to shorten the transient period.

(vi) The initial growth rate estimated in Eq. (6) is usually much higher than the simulation results. Estimates of stability threshold made by using Eq. (7) appear to be too conservative in general.

As an example, we focus our study here on a few specific density profiles and initial conditions with Δ_p and proton intensity chosen near the stability threshold. Thus, we consider a constant λ_e and four types of λ_p: constant, elliptical, parabolic, and quartic (parabolic squared). Four initial perturbations on the proton centroid are investigated: wavelength proportional to $\sqrt{\lambda_p}$, noise, 100 MHz, and 250 MHz. All four initial perturbations have a same constant envelope. The electron centroid was assumed to be unperturbed when entering the proton bunch. The following PSR parameter values were used for computation: $\gamma = 1.85$, $a = 1.5$ cm, the circumference $C = 90$ m, 2.74×10^{13}

protons per bunch, $\omega_\beta = 40$ MHz, and $L/v = 200$ ns (for a short bunch). We assume $\Delta_p = 0.125\%\omega_\beta$, $\Delta_e = 1.25\omega_\beta$, and a flat amount of electrons corresponding to a 2% of charge neutralization in the case of constant proton line density. For the four types of λ_p considered here, the peak electron bounce frequency in the proton bunch is between 100 MHz and 200 MHz.

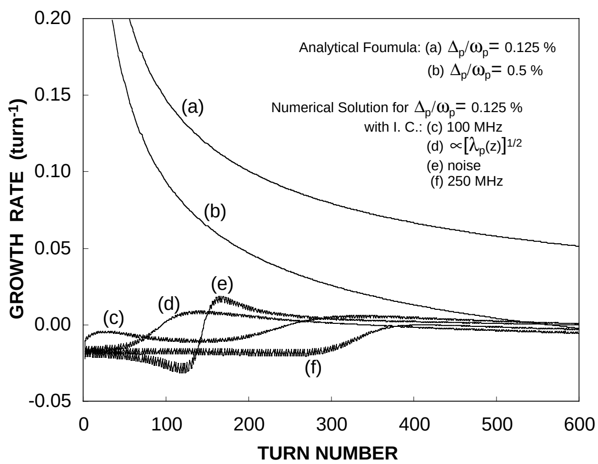

Figure 1: The growth rate at the tail of the proton bunch as a function of the turn number in PSR for different initial conditions and the growth rate computed using Eq. (6). Parabolic proton line density is considered here.

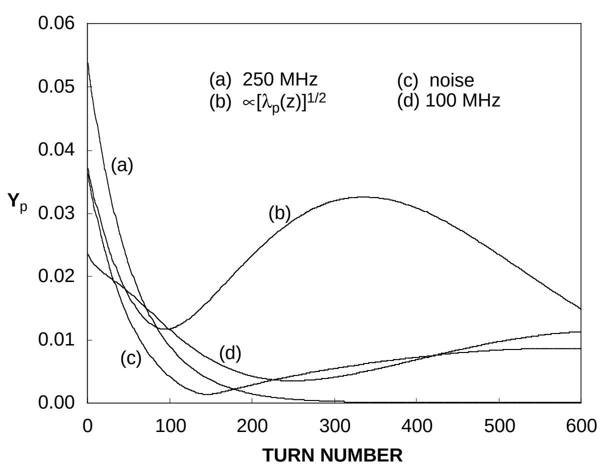

Figure 2: The amplitude of oscillation for the proton centroid is shown as a function of the turn number in PSR for the cases considered in Fig. 1. The ordinate has an arbitrary unit.

Shown in Fig. 1 is the growth rate at the tail of the proton bunch as a function of the turn number in PSR for different initial conditions and parabolic proton line density. The growth rate computed using Eq. (6) is also shown for comparison. The corresponding amplitude of proton centroid oscillation is shown in Fig. 2. Figure 3 shows the growth rate at the tail of the proton bunch as a function of the turn number for different proton line densities and a same noise initial condition. It is seen that the initial perturbation with wavelength proportional to $\sqrt{\lambda_p}$ is the least stable one among the four initial conditions considered.

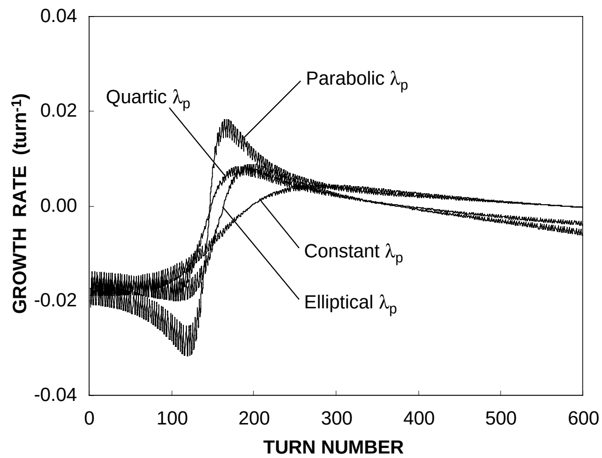

Figure 3: The growth rate at the tail of the proton bunch as a function of the turn number in PSR for different proton line densities and a same noise initial condition.

CONCLUSIONS

We have carried out a numerical study of the transverse e-p instability using the centroid equations derived from Lorentzian distribution of particles' oscillation frequencies and the model of one-pass interaction between the stationary electrons and traveling proton bunch. Numerical results were compared with the asymptotic solution. Good qualitative agreements were found when the initial perturbation in the numerical solution is evolved into the asymptotic regime at large time. The initial evolution of perturbations was investigated for various proton line densities and initial conditions of the proton bunch. Some qualitative understanding has been acquired from these numerical solutions. Notably, we found that the asymptotic solution tend to overestimate the growth rate and the frequency spread of proton oscillation required for stability in general. Thus, unlike the situation in analyzing a usual beam stability, the extrapolation of the growth rate from the asymptotic regime to the initial state is much more restricted for the e-p instability in a bunched beam of non-uniform line density. We also found that the perturbations with frequency below the peak electron bounce frequency have stronger influence on the initial growth rate than the high-frequency perturbations.

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