SINGLE-MODE COHERENT SYNCHROTRON RADIATION INSTABILITY

S. Heifets and G. Stupakov
Stanford Linear Accelerator Center, Stanford University, Stanford, CA 94309

INTRODUCTION

A relativistic electron beam moving in a circular orbit in free space can radiate coherently if the wavelength of the synchrotron radiation exceeds the length of the bunch. In accelerators coherent radiation of the bunch is usually suppressed by the screening effect of the conducting walls of the vacuum chamber [1, 2, 3]. The screening effect is much less effective for short wavelengths, but if the wavelength is shorter than the length of the bunch (assuming a smooth beam profile), the coherent radiation becomes exponentially small. However, an initial density fluctuation with a characteristic length much shorter than the screening threshold would radiate coherently. If the radiation reaction force is directed so that it drives the growth of the initial fluctuation, one can expect an instability that leads to micro-bunching of the beam and an increased coherent radiation at short wavelengths.

In Ref. [6], the growth rate of the beam instability driven by the coherent synchrotron radiation (CSR) was found using the so called “CSR impedance” [4, 5] that neglects the shielding effect of the walls and assumes a continuous spectrum of radiation. In many cases, the instability is limited to relatively long wavelengths, and it may be affected by the wall shielding effect [1]. Close to the shielding threshold, one has to take into account that the spectrum of the synchronous modes of radiation is discrete, and the instability may be driven by a single synchronous mode rather than a continuous spectrum.

In this paper we study a linear regime of single-mode CSR instability. As in Ref. [6], we assume that the bunch is much longer than the wavelength of the modulation and consider a coasting beam model. The nonlinear regime of the instability is described in accompanying paper [7].

SYNCHRONOUS MODES IN TOROIDAL BEAM PIPE CLOSE TO SHIELDING THRESHOLD

A relativistic beam moving in a toroidal beam pipe interacts with synchronous modes that have phase velocity equal to the speed of light. For a perfectly conducting walls of the toroid, those modes have discrete frequencies. Such modes have been extensively studied in the past [3, 8]. Recently, a new approach to the problem [9] extended the previous analysis and allowed to treat arbitrary cross sections of the toroid.

Following Ref. [9], we assume that the characteristic size of the pipe cross section a is much smaller than the toroid radius R, so that the ratio $\sqrt{a/R}$ is a small parameter. For a given toroid, the synchronous modes have wavenumbers k greater than a minimal value $k_{\min} = \omega_{\min}/c$:

$$k \geq \frac{\omega_{\min}}{c} \sim \frac{R^{1/2}}{a^{3/2}} \gg a^{-1} .$$

Each mode is characterized by its frequency ω_n, the wavenumber $q_n = \omega_n/c$, and the group velocity v_{gn}. The wake of each mode is

$$w_n(z) = 2\chi_n \cos\left(q_n z\right) , \tag{1}$$

where χ_n is the loss factor. The total wake is the sum of partial contributions of all modes: $w(s) = \sum_n w_n(s)$.

The lowest synchronous mode wavenumber is of order of k_0, where

$$k_0 = \frac{\pi}{a}\sqrt{\frac{R}{a}} .$$

For example, for a beam pipe of a square cross-section with the side a, $k_{\min} = 1.52\, k_0$. The loss factor per unit length χ_1 and the group velocity v_{g1} for this mode are $\chi_1 = 4.94/a^2$, $1 - v_{g1}/c = 0.62\, a/R$. Note that such modes propagate with the group velocity close to the speed of light. The next mode with a nonzero loss factor has a frequency $\omega_2 = 2.79\, ck_0$ and the loss factor $\chi_2 = 3.01/a^2$. We emphasize here that the distance between the synchronous modes in the vicinity of $\omega_{\min}$ is of the order of their frequency, and in that sense the modes are well separated on the frequency scale. Similar results hold for the round toroidal pipe [9].

INTERACTION OF THE BEAM WITH A SINGLE SYNCHRONOUS MODE IN LINEAR APPROXIMATION

The interaction of the beam with electromagnetic waves is usually described in terms of the beam impedance (see, *e.g.*, [10]). For discrete synchronous modes, the beam impedance has singularities centered at the mode frequencies. In this case, a direct application of the standard approach may give an incorrect result. In Ref. [11], a derivation of equations for beam-wave interaction is given based on the Maxwell-Vlasov system of equations without using the concept of the impedance. In this section, we obtain the equations of Ref. [11] using a simple heuristic argument that “fixes” the conventional approach by taking into account the effect of retardation.

We use a one dimensional model for the beam, neglecting effect of the finite transverse emittance and considering a distribution function $f(z, \delta, t)$, where z is the longitudinal coordinate measured from a reference particle moving with the speed of light, and δ is the energy offset relative to the nominal energy E_0, $\delta = (E - E_0)/E_0$. We also assume that the modulation wavelength is small compared to the bunch length and consider a coasting beam with the linear density n_b equal to the local linear density of the bunch.

In the linear approximation, the perturbation due to the electromagnetic field can be considered as small: $f = f_0(\delta) + f_1(z, \delta, t)$, with $|f_1| \ll f_0$. The linearized Vlasov equation for f_1 is

$$\frac{\partial f_1}{\partial t} - \eta c\delta \frac{\partial f_1}{\partial z} + \frac{e}{\gamma mc}\mathcal{E}(z,t)\frac{\partial f_0}{\partial \delta} = 0\,, \qquad (2)$$

where η is the momentum compaction factor, γmc^2 is the nominal beam energy, and $\mathcal{E}(z, t)$ is the longitudinal component of the electric field. The function f is normalized so that $\int f dz d\delta$ gives the number of particles in the beam.

The usual formula for the electric field in terms of the wake function is [10]:

$$\mathcal{E}(z,t) = -e\int_z^\infty dz' \int d\delta\, w(z'-z) f_1(z', \delta, t)\,. \qquad (3)$$

However, it misses an important effect of the wake retardation that we need to take into consideration here. Indeed, the wave radiated at position s' at time t' and propagating in the forward direction to s, such that $s > s'$, will take time $t - t' = (s - s')/v_g$ to arrive at the destination, where v_g is the group velocity of the wave. Since $s' = z' + ct'$ and $s = z + ct$ we find from the above relation the retardation time between the emission and arrival in terms of coordinate z: $t - t' = (z' - z)/(c - v_g)$. To include the effect of the retardation in Eq. (3), we need to take the distribution function in Eq. (3) at the time of emission of the wave:

$$\begin{aligned}\mathcal{E}(z,t) &= -e\int_z^\infty dz' \int d\delta\, w(z'-z) \\ &\times\; f_1\left(z', \delta, t - \frac{z'-z}{c-v_g}\right)\,.\end{aligned}$$

This equation replaces Eq. (3) in our derivation. Contrary to the usual case of the geometric impedance, where the group velocity is small, effect of retardation here is important because v_g is close to the speed of light.

Note, that for the free space CSR, the retardation time is equal to $[24R^2(z' - z)]^{1/3}$ defined by the difference of the path length along the circle for the beam and the straight line for the radiation. A more detailed study of the retardation effect for the CSR wake in vacuum can be found in Ref. [12].

For what follows, it is convenient to introduce the Fourier transform g_1 of the perturbation of the distribution function $g_1(\omega, q, \delta) = \int dtdz\, e^{i(\omega t - qz)} f_1(z, \delta, t)$. It follows from Eq. (2):

$$g_1(\omega,q,\delta) = -\frac{ie}{\gamma mc}\frac{E(\omega,q)}{\omega+\eta c\delta q}\frac{\partial f_0}{\partial\delta}\,, \qquad (4)$$

where $E(\omega, q) = \int dtdz\, e^{i(\omega t - qz)}\mathcal{E}(z, t)$. The quantity $E(\omega, q)$ can be found by Fourier transforming Eq. (4) and using the wake from Eq. (1):

$$E(\omega,q) = \sum_n \frac{-ie\chi_n(c-v_{gn})}{\omega+(c-v_{gn})(q-q_n)}\int d\delta g_1(\omega,q,\delta)\,. \qquad (5)$$

To obtain the above equation, we assumed that the frequency $|\omega| \sim (1-\beta_g)|q-q_n| \ll \omega_n$, which is equivalent to using only the synchronous part of the wake: $\cos(q_n z) \to e^{-iq_n z}/2$. Combining Eqs. (4) and (5) yields the dispersion relation

$$1 = -\sum_n \frac{\lambda_n}{\omega/c + (1-\beta_{gn})\Delta q_n}\int d\delta \frac{\partial f_0/\partial\delta}{\omega+\eta cq\delta}\,, \qquad (6)$$

where $\lambda_n = r_e c(1 - \beta_{gn})\chi_n/\gamma$, $\Delta q_n = q - q_n$, with $r_e = e^2/mc^2$. In Eq. (6) we took into account that $v_{gn} \approx c$. As always in stability theory, the integration in Eq. (6) goes in the complex plane above the pole $\delta = -\omega/\eta cq$. For a real value of q, Eq. (6) defines a complex frequency ω the imaginary part of which gives the growth rate of the instability. Alternatively, we can consider real ω and find a complex wavenumber q describing a periodic perturbation growing or decaying along the beam pipe.

Note that the frequency of the mode Ω observed in the laboratory frame, where it has a dependence $e^{i(qs-\Omega t)}$, is equal to $\Omega = \omega + qc$.

DISPERSION RELATION FOR A SINGLE MODE

In the single-mode approximation, we leave only one term in the dispersion equation Eq. (6) corresponding to the lowest synchronous mode with frequency ω_n and $q_n = \omega_n/c$. Let us assume that the distribution function $f_0(\delta)$ is Gaussian with the rms energy spread δ_0, $f_0 = (n_b/\delta_0)\rho_0(\delta/\delta_0)$ with $\rho_0(\xi) = e^{-\xi^2/2}/\sqrt{2\pi}$. Eq. (6) takes the form

$$\frac{\omega}{c} - (1-\beta_{gn})\Delta q_n = -\frac{n_b\lambda_n}{\eta\omega_n\delta_0^2}\int \frac{d\xi \frac{d\rho_0}{d\xi}}{\frac{\omega}{\eta\omega_n\delta_0}+\xi}\,, \qquad (7)$$

where we replaced q under the integral by q_n and used $q_n = \omega_n/c$. Depending on the ratio $\omega/\eta\omega_n\delta_0$, there are two possible regimes for the instability: a large energy spread regime, when $|\omega| \ll |\eta\omega_n\delta_0|$, and a "cold beam" approximation when the opposite inequality holds. We consider here the latter case only, as a more relevant to the parameters of the existing accelerators (see below). In this case, we can evaluate the integrand in Eq. (7) asymptotically in the limit $|\omega/\eta\omega_n\delta_0| \gg 1$, which results in the cubic dispersion equation:

$$\omega^2\left[\frac{\omega}{c} - (1-\beta_{gn})\Delta q_n\right] = -n_b\lambda_n\eta\omega_n\,. \qquad (8)$$

For $\Delta q_n = 0$, one of the roots has a positive imaginary part:

$$\omega = \mu\, e^{i\pi/3}\,, \qquad (9)$$

where we introduced the parameter μ

$$\mu = (n_b \lambda_n c \eta \omega_n)^{1/3} = c \left[\frac{r_e n_b \omega_n \eta \chi_n}{c\gamma(1-\beta_{gn})} \right]^{1/3} .$$

Note that for a cold beam there is no threshold for the instability. The estimate of the integral term in the dispersion equation used above neglects the Landau damping and is valid provided $|\mu| \gg \eta\omega_n\delta_0$.

For a general case of arbitrary detuning Δq_n, Eq. (8) can be written in the dimensionless form as

$$x^2(x+y) + 1 = 0 \,, \qquad (10)$$

by introducing $x = \omega/\mu$, $y = c\Delta q_n(1-\beta_{gn})/\mu$. Eq. (10) can be easily solved numerically—it has three roots one of which corresponds to the instability. The maximum growth rate is achieved at zero detuning, $\Delta q_n = 0$ and is equal to $\mathrm{Im}\,\omega = \sqrt{3}/2\mu$.

Table 1 gives parameters and compares the growth rate for four accelerators: the Low Energy Ring (LER) and the High Energy Ring (HER) of PEP-II accelerator at SLAC, Advanced Light Source at the Berkeley National Laboratory, and the VUV ring at the National Synchrotron Light Source at BNL. For the ALS, we used beam parameters for the regime in which bursts of infrared radiation were observed [13]. Calculations were made for the lowest synchronous mode (which frequency is denoted by ω_1) assuming a square cross section of the vacuum chamber with the size a equal to the vertical full gap of the beam pipe. Since the real shape of the cross section usually differs from the square, the results in the table should be considered as a rough estimate of the instability parameters. For the linear density of the beam n_b, we used the quantity $N_p/\sqrt{2\pi}\sigma_z$, which gives the maximum linear density in a gaussian bunch (N_p is the number of particles in the bunch, σ_z is the rms bunch length). Note that the ratio $\mu/\eta\omega_1\delta_0$ in the last line of the table related the cold beam approximation—it is large in all cases except for the HER PEP-II where it is close to one.

DISCUSSION

The model developed in this paper considers a ring as a perfect toroid with a constant bending radius. We derived equations for the beam-mode interaction, found the growth rate for the instability, and estimated it for several machines. We have shown that in this case, due to the persistent interaction with a resonant mode, the beam becomes unstable even at low currents.

In real lattice, bending magnets are usually separated by straight sections, which also have a different cross section of the vacuum chamber. The beam-mode interaction ceases in the straight sections and the amplitude and phase of the mode will most likely change after the passage through the straights. The beam density modulation induced in one bend, after passage through a straight section, will serve as a seed for the instability in the next one. We expect that in a real lattice the instability would develop with a growth rate smaller than in an ideal toroid. A study of this case will be published in a separate paper.

Table 1: Parameters relevant to the instability for PEP-II low energy (LER) and high energy (HER) rings, ALS, and VUV NSLS ring.

Parameter	LER	HER	ALS	VUV NSLS
Energy, GeV	3.1	9.0	1.5	0.81
η, 10^{-3}	1.3	2.1	1.4	2.4
δ_0, 10^{-4}	8.1	6.1	7.1	5.0
n_b, 10^{10} cm^{-1}	3.7	0.82	7	3.6
a, cm	5	5	4	4.2
R, m	13.7	165.0	4.0	1.9
$\omega_1/2\pi$, GHz	75.5	260	57	36.6
χ, V/pC/m	18	18	28	25
μ, 10^6 s^{-1}	7.5	2.5	18	22
n_{cr}, 10^{10} cm^{-1}	13	140	3	0.8
$\mu/(\eta\omega_1\delta_0)$	15	1.2	84	50

ACKNOWLEDGEMENTS

We thank S. Krinsky and Z. Huang for usefull discussions. This work was supported by the Department of Energy, contract DE–AC03–76SF00515.

REFERENCES

[1] J. Schwinger, *On radiation by electrons in a betatron* (1945).

[2] L. Schiff, Rev. Sci. Instr. **17**, 6 (1946).

[3] R. L. Warnock and P. Morton, Part. Accel. **25**, 113 (1990).

[4] J. B. Murphy, S. Krinsky, and R. L. Gluckstern, in *Proc. of PAC 1995*.

[5] Y. S. Derbenev, et al. DESY FEL Report TESLA-FEL 95-05, September 1995.

[6] S. Heifets and Stupakov G. V., Phys. Rev. ST Accel. Beams **5**, 054402 (2002).

[7] S. Heifets and G. Stupakov, in *PAC03* (2003).

[8] K.-Y. Ng, Part. Accel **25**, 153 (1990).

[9] G. V. Stupakov and I. A. Kotelnikov, Phys. Rev. ST Accel. Beams **6**, 034401 (2003).

[10] A. W. Chao, *Physics of Collective Beam Instabilities in High Energy Accelerators* (Wiley, New York, 1993).

[11] S. Heifets and G. Stupakov, Preprint SLAC-PUB-9627, SLAC (2003).

[12] S. Heifets, Preprint SLAC-PUB-9054, SLAC (2001).

[13] J. Byrd, et al. Phys. Rev. Lett. **89**, 224801 (2002).