NONLINEAR REGIME OF A SINGLE-MODE CSR INSTABILITY

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INTRODUCTION

In Ref. [1] the growth rate of the beam instability driven by the coherent synchrotron radiation (CSR) was found using the so called “CSR impedance” [2, 3] that neglects the shielding effect of the walls and assumes a continuous spectrum of radiation. In many cases, the instability is limited to relatively long wavelengths where it may be affected by the wall shielding effect [4]. Close to the shielding threshold, one has to take into account that the spectrum of synchronous modes of radiation is discrete, and the instability may be driven by a single mode rather than a continuous spectrum.

The linear theory of single-mode CSR instability is developed in Refs. [1, 5]. In this paper, we study nonlinear regime of the instability. As in Ref. [1], we assume that the bunch is much longer than the wavelength of the modulation and consider a coasting beam model.

NONLINEAR REGIME OF THE INSTABILITY

In Refs. [1, 5] we calculated the growth rate for a single-mode instability as a function of detuning $q - q_n$, where q is the wavenumber of the perturbation and q_n is the wavenumber of the nth synchronous mode in a toroidal waveguide. The growth rate is localized in a small vicinity of q_n with a maximum at $q = q_n$.

When the amplitude of the unstable mode becomes large, the linear theory is not valid any more and one has to use the full Vlasov equation for the distribution function $f(z, \delta, t)$:

$$\frac{\partial f}{\partial t} - \eta c \delta \frac{\partial f}{\partial z} + \frac{e}{\gamma mc} \mathcal{E}(z,t) \frac{\partial f}{\partial \delta} = 0 . \qquad (1)$$

Here z is the longitudinal coordinate measured relative to a reference particle moving with the speed of light, δ is the energy offset relative to the nominal energy E_0, $\delta = (E - E_0)/E_0$, η is the momentum compaction factor, γmc^2 is the nominal beam energy, and $\mathcal{E}(z,t)$ is the longitudinal component of the electric field. The function f is normalized so that $\int f dz d\delta$ gives the number of particles in the beam.

An important approximation that we make in the nonlinear regime is that the evolution of the instability is governed by a single mode with a wavenumber q. One would expect that this wavenumber is equal to q_n—the mode that has the maximum growth rate in the linear regime—however, for the sake of generality, we treat q as arbitrary (but close to q_n). The derivation of the equation for $\mathcal{E}(z,t)$ describing the interaction of the beam with the mode is given in Ref. [6]. Together with Eq. (1), they constitute a system that describes nonlinear evolution of the beam with the single mode of the field. Here we formulate this system of equation without derivation.

It is convenient to introduce dimensionless variables τ, ζ, and p instead of t, z and δ, respectively, where

$$\tau = \mu t, \quad \zeta = qz, \quad p = -\frac{\eta\omega_n}{\mu}\delta ,$$

and

$$\mu = c\left[\frac{r_e n_b \omega_n \eta \chi_n}{c\gamma(1-\beta_{gn})}\right]^{1/3} ,$$

where ω_n, χ_n and $c\beta_{gn}$ are the frequency, loss factor and the group velocity of the synchronous mode, respectively, ($\omega_n = cq_n$), and $r_e = e^2/mc^2$. We introduce the amplitude $A(\tau)$ such that,

$$\mathcal{E} = -\frac{\gamma mc\mu}{e\eta\omega_n}\left[A(\tau)e^{iqz} + \text{c.c.}\right] ,$$

and the dimensionless distribution function

$$F(\zeta, p, \tau) = \frac{1}{2\pi n_b}\frac{\mu}{\eta\omega_n} f ,$$

normalized by the condition $\int_{-\infty}^{\infty} dp \int_0^{2\pi} d\zeta F(\zeta, p, \tau) = 1$. In these variables, the beam dynamics is described by the following equation,

$$\frac{\partial F}{\partial \tau} + p\frac{\partial F}{\partial \zeta} + \left[A(\tau)e^{i\zeta} + \text{c.c.}\right]\frac{\partial F}{\partial p} = 0 , \qquad (2)$$

and the amplitude $A(\tau)$ satisfies the equation

$$\frac{\partial A(\tau)}{\partial \tau} = \langle e^{-i\zeta}\rangle + iuA , \qquad (3)$$

with

$$\begin{aligned} \langle e^{-i\zeta}\rangle &= \int_{-\infty}^{\infty} dp \int_0^{2\pi} d\zeta F(\zeta, p, \tau)\, e^{-i\zeta} , \\ u &= \frac{c}{\mu}(q - q_n)(1 - \beta_{gn}) . \end{aligned} \qquad (4)$$

Characteristics of Eq. (2) are equations of motion for a single particle:

$$\frac{d\zeta}{d\tau} = p, \quad \frac{dp}{d\tau} = [A(\tau)e^{i\zeta} + \text{c.c.}] . \qquad (5)$$

Eq. (2), (3) and Eq. (5) constitute a full system of equations. These equations have an integral of motion:

$$C = |A|^2 - \langle p \rangle \,, \tag{6}$$

which reflects conservation of energy—the sum of the wave energy and the beam energy is constant during the interaction.

The system of equations (2), (3), and Eq. (5) is encountered in other problems of nonlinear beam-wave interaction, *e.g.*, in the one-dimensional FEL theory [7, 8], with the parameter μ being equivalent to the Pierce parameter ρ. The solution of the system on a limited time interval can be obtained by numerical methods. In the numerical approach, the beam is represented by a finite number M of macroparticles, and the average $\langle e^{i\zeta} \rangle$ is approximated by the sum $(1/M)\sum_1^M e^{-i\zeta_k}$ over all particles' coordinates ζ_k. The result of such a solution—the absolute value $|A|$ of the amplitude of the wave—is shown in Fig. (1). The amplitude of an initial small perturbation saturates after an initial exponential growth and exhibits oscillations at frequency of the order of the bounce frequency of particles in the bucket of the excited wave. Fig. 1 agrees with a similar solution obtained earlier in Ref. [7].

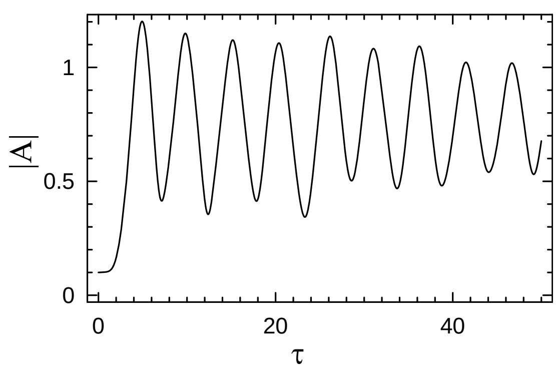

Figure 1: The dependence of the amplitude $|A|$ versus τ in the nonlinear regime of the instability.

SYNCHROTRON DAMPING AND QUANTUM DIFFUSION

Contrary to the FEL theory, where it usually suffices to track the solution on several gain lengths only, for a beam in the storage ring we may be interested in time comparable to the synchrotron damping time. The analysis in this case has to include the synchrotron damping and diffusion due to quantum fluctuations effects. One of the difficulties of such analysis is that the damping time typically is larger than the synchrotron oscillation period in the damping right so that one has also take into account synchrotron oscillations of a particle in the bunch. Here, however, we will consider an idealized formulation which neglects synchrotron oscillations, but includes synchrotron damping and diffusion due to quantum fluctuations in synchrotron radiation. A more detailed study, with account of synchrotron motion, can be found in Ref. [9].

To include the effects of synchrotron damping and quantum diffusion into the interaction of the wave with the beam, we need to use the Vlasov-Fokker-Planck equation [10]. In our dimensionless variables it has the following form

$$\begin{aligned}&\frac{\partial F}{\partial \tau} + p\frac{\partial F}{\partial \zeta} + \left[A(\tau)e^{i\zeta} + \text{c.c.}\right]\frac{\partial F}{\partial p}\\ &=\Gamma\frac{\partial}{\partial p}\left(\Delta^2\frac{\partial F}{\partial p} + pF\right)\,,\end{aligned}$$

where Γ and Δ are related to the synchrotron radiation damping $\gamma_{\rm SR}$ and the rms energy spread $\delta_{\rm SR}$ due to the quantum fluctuations in the synchrotron radiation:

$$\Gamma = \frac{\gamma_{\rm SR}}{\mu}\,, \quad \Delta = \frac{\eta\omega_n\delta_{\rm SR}}{\mu}\,.$$

Note that with damping the integral C in Eq. (6) is not conserved any more: $\frac{d}{d\tau}\left(|A|^2 - \langle p \rangle\right) = \Gamma\langle p \rangle$ instead of Eq. (6).

In order to carry out numerical simulation of the Vlasov-Fokker-Plank equation, we note that this equation is equivalent to a set of single-particle equations of motion with damping and an external force $\kappa(\tau)$:

$$\frac{d\zeta}{d\tau} = p, \quad \frac{dp}{d\tau} = [A(\tau)e^{i\zeta} + \text{c.c.}] - \Gamma p + \kappa(\tau)\,.$$

where $\kappa(\tau)$ is a random function of time τ with zero average value $\langle\kappa\rangle = 0$ and the correlation function

$$\langle \kappa(\tau)\kappa(\tau')\rangle = 2\Gamma\Delta^2\,\delta(\tau - \tau')\,.$$

In our simulation, we used a discrete time mesh τ_i with the time step $\tau_s = \tau_{i+1} - \tau_i$ and a finite number of particles M. On each interval, we first solved the system of the differential equations Eqs. (5) and (3) without damping and fluctuations. The damping and fluctuations were taken into account at the end of each step by changing the variable p for each particle:

$$p_k \to p_k - \Gamma\tau_s p_k + \sqrt{24\tau_s\Gamma\Delta^2}\,\xi\,,$$

where ξ is a random number uniformly distributed in the range $[-1/2, 1/2]$. This algorithm was tested on the case without the wave, $A = 0$, and also for the case of an external wave with constant amplitude $A = \text{const}$, when the Vlasov-Fokker-Planck equation has analytical solutions. In both cases we found a good agreement between the numerical and analytical solutions.

The simulations were carried out for the parameters close to that of ALS: $\mu = 3.2\cdot 10^7$ s^{-1}, $\omega_n = 1.0\cdot 10^{12}$ s^{-1}, $\Delta = 0.032$. However, to speed up the tracking, we increased the parameter Γ from the ALS value $2.0\,10^{-6}$ to $2.0\,10^{-2}$. We expect that such a rescaling of Γ accelerates the manifestation of the synchrotron damping effects without qualitatively changing the solution. Typically we used from 200 to 800 particles in the simulation.

The results of the tracking for $\tau \approx 1000$ (corresponding to approximately 20 damping times) are shown in Fig. 2 and Fig. 3. Fig. 2 shows the amplitude $|A(\tau)|$, and

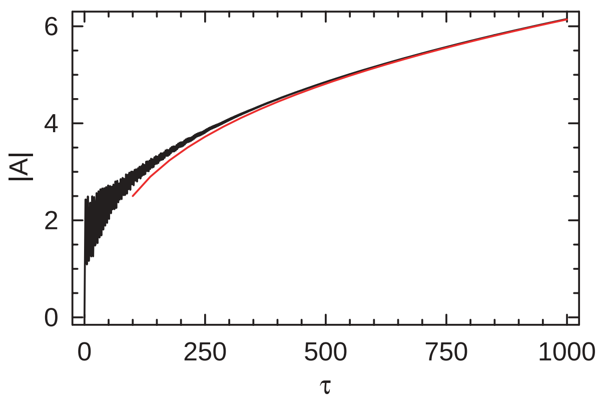

Figure 2: The absolute value of the amplitude $|A(\tau)|$ as a function of τ. Black curve shows the result of simulation, red curve–analytical solution of Eq. (8).

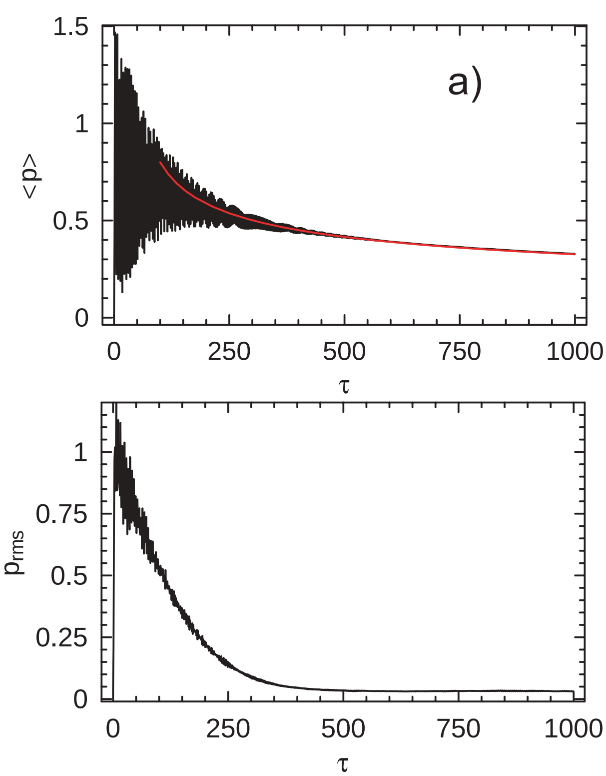

Figure 3: Numerical simulation of nonlinear regime of the instability: *a*)—the average momentum $\langle p\rangle$, *b*)—the rms momentum spread $\Delta p_{\rm rms}$. The red line shows the the result of the analytical model.

Fig. 3 shows the average over distribution function momentum $\langle p\rangle$ and the rms spread in p, $\Delta p_{\rm rms}$, as functions of time. For the time interval small compared with the damping time $\tau \lesssim 50$, results of tracking reproduce Fig. 1. For larger time intervals, $\tau \gg 50$, the amplitude $|A|$ keeps growing, and the beam comes to a quasi equilibrium, with a slowly changing values of $\langle p\rangle$ and $\Delta p_{\rm rms}$. Note also a relatively small value of $\Delta p_{\rm rms}$, which means that particles of the beam are well localized in the p-space.

The numerical results shown in Figs. 2 and 3 give us an indication of an analytical solution to the problem in the limit of large τ. In this solution we assume that

$$A(\tau) = \frac{1}{2} i A_0(\tau) e^{-i\nu(\tau)\tau} , \qquad (7)$$

where the function $A_0(\tau)$ and frequency $\nu(\tau)$ are slow functions of time. Particles are trapped by the wave $e^{i(\zeta-\nu\tau)}$ and drift with the rate $d\zeta/d\tau = \nu$. It can be shown [6] that the amplitude $A_0(\tau)$ grows in time due to damping with the rate

$$\frac{dA_0}{d\tau} = \frac{2}{\sqrt{1 + A_0^4/(2\Gamma)^2}} . \qquad (8)$$

Since this equation determines asymptotic behavior of A in the limit $\tau \to \infty$, the initial condition for it is not defined. For the purpose of comparison with the numerical solution, we considered an initial condition $A(\tau_0) = A_*$, with A_* as a fitting parameter. The result of integration of Eq. (8) with $A(100) = 2.5$ is shown in Fig. 2 in red color, in good agreement with the numerical solution. It is straightforward to show that for large τ it follows from Eq. (8) that $A_0 \propto \tau^{1/3}$. The averaged momentum of the particles $\langle p\rangle$ in this model can also be found ans is shown as a red line in Fig. 3a.

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REFERENCES

[1] S. Heifets and Stupakov G. V., Phys. Rev. ST Accel. Beams **5**, 054402 (2002).

[2] J. B. Murphy, S. Krinsky, and R. L. Gluckstern, in *Proc. of PAC 1995*.

[3] Y. S. Derbenev, et al. DESY FEL Report TESLA-FEL 95-05, (September 1995).

[4] J. Schwinger, (1945), unpublished.

[5] S. Heifets and G. Stupakov, in *PAC03* (2003).

[6] S. Heifets and G. Stupakov, Preprint SLAC-PUB-9627, SLAC (2003).

[7] R. Gluckstern, S. Krinsky, and H. Okamoto, Phys. Rev. **E47**, 4412 (1993).

[8] N. A. Vinokurov, et al. NIM **A 475**, 74 (2001).

[9] S. Heifets, Preprint SLAC-PUB-9626, SLAC (2003).

[10] A. Piwinski, in *Proc. CERN Accelerator School, Gif-sur-Yvette, 1984* no. 85-19 in CERN Yellow Report, pp. 432–462, (1985).