A LIE TRANSFORM PERTURBATION SCHEME FOR HAMILTONIAN AVERAGING IN SELF CONSISTENT SYSTEMS

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Abstract

A periodic focusing system is reduced to an equivalent continuous focusing one for a beam with space charge by averaging over the lattice oscillations. The Lie transform perturbation method is used to canonically transform the laboratory phase space variables to slowly oscillating variables. A similar averaging over the lattice period was performed by R.C. Davidson, H. Qin and P.J. Channell [Phys Rev ST 2, 074401 (1999)] using the Poincare-Von Ziepel perturbation method. The Lie transform method offers certain advantages in that it retains the original form of the Hamiltonian before beginning the process of canonical transformation to a slowly oscillating coordinate frame. On the other hand, the Poincare-Von Ziepel method requires one to make a Taylor expansion of the Hamiltonian in terms of the as yet undetermined expansion terms of the transformed phase space variables. The Lie transform method avoids such a Taylor expansion and so the formulation is less tedious. It will be demonstrated that performing the reverse transformation to the original phase space variables is also straight forward in the Lie transform method.

INTRODUCTION

In many applications of dynamical systems, one is primarily interested in the long time behavior compared to certain fast time scales over which the system evolves. Hamiltonian averaging techniques have been effective in obtaining a set of equations that contain only long-time processes and retain the effects produced by the short time scale processes only up to a desired approximation. The standard procedure is to perform a perturbation canonical transformation to a slowly oscillating reference frame. The Lie Transform perturbation method is more convenient for such a procedure when compared to methods based on the Hamilton-Jacobi transformation. A Lie transformation can be expressed in Poisson bracket notation as an analogy to Hamilton's equation with respect to a continuously varying parameter representing "Time" and a Lie generating function representing the "Hamiltonian". This enables one to develop the whole formulation in Poisson brackets form making inverse transformations at every intermediate stage unnecessary because the results of Poisson brackets are invariant under canonical transformations.

TRANSFORMATION EQUATIONS IN TERMS OF PERTURBATION EXPANSIONS

A Lie Transformation is defined in terms of a Lie generating function w which satisfies the Poisson bracket relation,

$$\frac{dZ}{d\epsilon} = \{Z(z),\ w(z,t,\epsilon)\} \tag{1}$$

This describes a canonical transformation from the phase space vectors z to Z. The transformed variable, Z varies continuously with respect to the time like parameter ϵ with w being analogous to the Hamiltonian. The symplectic structure of the transformation guarantees that the transformation is canonical for all values of ϵ. A Lie operator is defined by $L = \{w,\ \}$. A transformation operator T transforms any function such that $Tf(z,t) = f(Z(z,t),t)$. T is equivalent to the "evolution" operator with respect to the "time" ϵ. For the identity function, this would simply be, $Tz = Z(z,t)$.

To obtain explicit equations for each perturbation term, every physical quantity and operator is expressed as a power series in ϵ known as the Deprit power series [3]. This would be $h(z,t,\epsilon) = \sum_{n=0}^{\infty} \epsilon^n h_n(z,t)$, $H(z,t,\epsilon) = \sum_{n=0}^{\infty} \epsilon^n H_n(z,t)$, $T(t,\epsilon) = \sum_{n=0}^{\infty} \epsilon^n T_n(t)$, $L(w) = \sum_{n=0}^{\infty} \epsilon^n L_n$, $w(z,t,\epsilon) = \sum_{n=0}^{\infty} \epsilon^n w_{n+1}(z,t)$. Where, h represents the original Hamiltonian and H represents the transformed Hamiltonian. Using these expansions, equations for each order of ϵ can be derived to determine the transformed Hamiltonian H with respect to w and h, and the transformation operator T with respect to the operator L. A rigorous derivation of these relationships can be found in Ref. [4] and a more brief one in Ref. [5]. These derivations are based on the work by Dewar [2]. In principle, the relationships can be derived up to any order. We give them here up to third order. In order to perform a time averaging, we need to set $h_0 = 0$ [7]. Using this, equations to determine the transformed Hamiltonian up to third order are,

$$H_0 = h_0 = 0 \tag{2}$$

$$\frac{\partial w_1}{\partial t} = H_1 - h_1 \tag{3}$$

$$\frac{\partial w_2}{\partial t} = 2(H_2 - h_2) - L_1(H_1 + h_1) \tag{4}$$

$$\begin{aligned}\frac{\partial w_3}{\partial t} &= 3(H_3 - h_3) - L_1(H_2 + h_2) \\ &\quad - L_2(H_1 + \frac{1}{2}h_1) - \frac{1}{2}L_1^2(h_1)\end{aligned} \tag{5}$$

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The expansion terms of the inverse transformation operator T^{-1} are given below up to third order. They are,

$$T_0^{-1} = I \tag{6}$$

$$T_1^{-1} = L_1 \tag{7}$$

$$T_2^{-1} = \frac{1}{2}L_2 + \frac{1}{2}L_1^2 \tag{8}$$

$$T_3^{-1} = \frac{1}{3}L_3 + \frac{1}{6}L_1L_2 + \frac{1}{3}L_2L_1 + \frac{1}{6}L_1^3 \tag{9}$$

APPLICATION TO AN INTENSE BEAM IN A LINEAR FOCUSING FIELD

The analysis in this section considers a thin, long nonneutral beam traveling with a constant axial velocity. The charged particles are subjected to a focusing force of the form $-[\kappa_x(s)x\hat{\mathbf{e_x}} + \kappa_y(s)y\hat{\mathbf{e_y}}]$. The axial distance s, is equivalent to time since the beam is assumed to be propagating at constant velocity. In normalized units (see for example Ref. [6]), the Hamiltonian for such a system in the transverse phase space (x, y, x', y') is given by

$$h = \frac{1}{2}(x'^2 + y'^2) + \frac{1}{2}\kappa_x(s)x^2 + \frac{1}{2}\kappa_y(s)y^2 + \psi(x, y, s) \tag{10}$$

The oscillation of the lattice function $\kappa(s)$ is considered to be much faster than the oscillation of the space charge potential $\psi(x, y, s)$. The aim here is to apply the perturbation theory of the previous section to average over the lattice oscillations. As mentioned in the previous section, we to set $h_0 = 0$. The Hamiltonian h is considered to be of the same order as the parameter ϵ. So, we have $h_1 = h$, $h_2 = 0$, and $h_3 = 0$. For this application of a charged particle beam, where the aim is to find an equivalent continuously focusing system, it is sufficient to carry out the procedure up to third order [1, 8].

The given Hamiltonian is now applied to Eqs. (2 - 5). In which everything is expressed in terms of the transformed variables. Equation (2) gives $H_0 = 0$, and Eq. (3) gives

$$\begin{aligned}\frac{\partial w_1}{\partial s} &= H_1 - \frac{1}{2}(X'^2 + Y'^2) \\ &-\frac{1}{2}\kappa_x(s)X^2 - \frac{1}{2}\kappa_y(s)Y^2 - \psi(X, Y, s)\end{aligned} \tag{11}$$

There are two unknown expressions in this equation. They are, H_1 and w_1 where H_1 needs to be chosen such that it retains only the slowly oscillating terms and cancels terms with a nonzero value when averaged over fast oscillations. This ensures that w_1 averages to zero over fast oscillations which is required in order for the perturbation scheme to be secular [4]. So,

$$H_1 = \frac{1}{2}(X'^2+Y'^2)+\frac{1}{2}\langle\kappa_x\rangle X^2+\frac{1}{2}\langle\kappa_y\rangle Y^2+\psi(X, Y, s), \tag{12}$$

where the angle brackets represent an average over a lattice period S. That is,

$$\langle ... \rangle = \frac{1}{S}\int_0^S ds(...) \tag{13}$$

Hereafter we assume that $\langle\kappa_{x,y}\rangle = 0$ which is true for most practical applications. Since the expansion terms of w appear in the form of derivatives in the transformation equations, it is sufficient to evaluate the indefinite integral with respect to s. Doing this for Eq. (11) gives,

$$w_1 = -\frac{1}{2}\kappa_x^{\mathrm{I}}(s)X^2 - \frac{1}{2}\kappa_y^{\mathrm{I}}(s)Y^2 \tag{14}$$

The Roman numerical superscripts represent an indefinite integral over s. Similarly, a superscript “II” will indicate a double integration over s and so on.

Moving now to the second order perturbation equation which is Eq. (4), we get

$$\frac{\partial w_2}{\partial s} = 2H_2 + 2\kappa_x^{\mathrm{I}}(s)XX' + 2\kappa_y^{\mathrm{I}}(s)YY' \tag{15}$$

Since both the terms on the right side are fast oscillating terms and average to zero, we need to set $H_2 = 0$. Integrating with respect to s yields

$$w_2 = 2(\kappa_x^{\mathrm{II}}(s)XX' + \kappa_y^{\mathrm{II}}(s)YY') \tag{16}$$

We now move to the third order equation which gives,

$$\begin{aligned}\frac{\partial w_3}{\partial s} &= 3(\kappa_x^{\mathrm{II}}(s)X\frac{\partial}{\partial X} + \kappa_y^{\mathrm{II}}(s)Y\frac{\partial}{\partial Y})\psi(X, Y, s) \\ &+3H_3 - 3\kappa_x^{\mathrm{II}}(s)X'^2 - 3\kappa_y^{\mathrm{II}}(s)Y'^2 \\ &+2\kappa_x^{\mathrm{II}}(s)\kappa_x(s)X^2 + 2\kappa_y^{\mathrm{II}}(s)\kappa_y(s)Y^2 \\ &-\frac{1}{2}(\kappa_x^{\mathrm{I}}(s))^2X^2 - \frac{1}{2}(\kappa_y^{\mathrm{I}}(s))^2Y^2\end{aligned} \tag{17}$$

Once again, H_3 needs to be chosen so that it cancels terms with nonzero averages over the lattice periods. The second term on the right side of Eq. (17) is a product between slow and fast oscillating terms. Up to the desired order, this product averages to zero over fast oscillations [7].

$$H_3 = \frac{1}{3}\langle\frac{1}{2}(\kappa_x^{\mathrm{I}})^2 - 2\kappa_x^{\mathrm{II}}\kappa_x\rangle X^2 + \frac{1}{3}\langle\frac{1}{2}(\kappa_y^{\mathrm{I}})^2 - 2\kappa_y^{\mathrm{II}}\kappa_y\rangle Y^2 \tag{18}$$

Up to third order, the transformed Hamiltonian given by $H = H_1 + H_2 + H_3$ represents an intense beam with continuous focusing. This can be expressed as

$$H = \frac{1}{2}(X'^2 + Y'^2) + \frac{1}{2}(\mathcal{K}_X X^2 + \mathcal{K}_Y Y^2) + \psi(X, Y, s) \tag{19}$$

where

$$\mathcal{K}_X = \frac{1}{3}\langle(\kappa_x^{\mathrm{I}})^2 - 4\kappa_x^{\mathrm{II}}\kappa_x\rangle \tag{20}$$

and

$$\mathcal{K}_Y = \frac{1}{3}\langle(\kappa_y^{\mathrm{I}})^2 - 4\kappa_y^{\mathrm{II}}\kappa_y\rangle \tag{21}$$

To determine w_3, we need to integrate Eq. (17). Retaining only terms of third order in ϵ, the integration yields

$$\begin{aligned}w_3 &= 3\kappa_x^{\mathrm{III}}(s)X\frac{\partial\psi}{\partial X} + 3\kappa_y^{\mathrm{III}}(s)Y\frac{\partial\psi}{\partial Y} \\ &-3\kappa_x^{\mathrm{III}}(s)X'^2 - 3\kappa_y^{\mathrm{III}}(s)Y'^2\end{aligned} \tag{22}$$

To transform back to the laboratory phase space variables, we make use of the operator T^{-1} given by Eqs.(6 - 9). The transformation will be performed up to third order. We may express the transformation in the form of a perturbation expansion as

$$x = T^{-1}X = X + x_1 + x_2 + ... \tag{23}$$

$$y = T^{-1}Y = Y + y_1 + y_2 + ... \tag{24}$$

$$x' = T^{-1}X' = X' + x'_1 + x'_2 + ... \tag{25}$$

$$y' = T^{-1}Y' = Y' + y'_1 + y'_2 + ... \tag{26}$$

The zeroth order terms X, Y, X', Y' represent the fact that $T_0^{-1} = I$, the identity transformation. The first order terms are obtained by operating T_1^{-1} which, from Eq. (7) is simply the operator L_1. So,

$$x_1 = \{w_1, X\} = 0 \tag{27}$$

$$y_1 = \{w_1, Y\} = 0 \tag{28}$$

$$x'_1 = \{w_1, X'\} = -\kappa_x^{\mathrm{I}}(s)X \tag{29}$$

$$y'_1 = \{w_1, Y'\} = -\kappa_y^{\mathrm{I}}(s)Y \tag{30}$$

Similarly, the second order terms can be got from Eq. (8) by operating T_2^{-1} on (X, X', Y, Y'). These are,

$$x_2 = \frac{1}{2}(L_2 + L_1^2)X = -\kappa_x^{\mathrm{II}}(s)X \tag{31}$$

$$y_2 = \frac{1}{2}(L_2 + L_1^2)Y = -\kappa_x^{\mathrm{II}}(s)X \tag{32}$$

$$x'_2 = \frac{1}{2}(L_2 + L_1^2)X' = \kappa_x^{\mathrm{II}}(s)X' \tag{33}$$

$$y'_2 = \frac{1}{2}(L_2 + L_1^2)Y' = \kappa_y^{\mathrm{II}}(s)Y' \tag{34}$$

The third order terms will be determined by the operator T_3^{-1} given by Eq. (9). This gives,

$$x_3 = 2\kappa_x^{\mathrm{III}}X' \tag{35}$$

$$y_3 = 2\kappa_y^{\mathrm{III}}Y' \tag{36}$$

$$x'_3 = \kappa_x^{\mathrm{III}}\frac{\partial\psi}{\partial X} + \kappa_x^{\mathrm{III}}X\frac{\partial^2\psi}{\partial X^2} \tag{37}$$

$$y'_3 = \kappa_y^{\mathrm{III}}\frac{\partial\psi}{\partial Y} + \kappa_y^{\mathrm{III}}Y\frac{\partial^2\psi}{\partial Y^2} \tag{38}$$

The expressions for w_1, w_2 and w_3 have been made use of in the form of Poisson brackets. They carry the transformed variables when performing an inverse transformation. Since Poisson brackets are canonically invariant, w could carry either variables depending on the operation that is being performed.

In this section, we have reduced a general linear focusing system to a continuous focusing system for an intense beam by canonically transforming to a slowly oscillating reference frame. The equivalent continuous focusing system can offer a variety of advantages. It can be solved more efficiently numerically because it can allow larger time steps. The transformed system can also also have symmetries that can reduce its dimensionality, for example, angular momentum is conserved for an alternating gradient focusing system with an azimuthally symmetric charge distribution. The averaged Hamiltonian is time independent, and so it can allow self consistent equilibrium where the phase space density is a function of the transformed Hamiltonian. That is, $F_0(X, Y, X', Y') = F(K)$. This would be a near equilibrium solution in the laboratory frame. In this section, relationships were also derived to transform the system back to the laboratory frame after performing the required calculations in the transformed reference frame,

SUMMARY

This paper demonstrates the use of the Lie transformation perturbation theory to perform a time averaging over fast oscillations for a beam with space charge in a periodic focusing channel. This derivation was previously done using the Poincare - Von Ziepel method Ref [1]. This derivation has the advantage that it does not assume the space charge to satisfy any specific equation like Poisson's or Maxwell's equation unlike those in Refs. [8, 9, 7]. In addition to that, the transformed Hamiltonian and the inverse transformation of phase space variables can be obtained in explicit form. The Lie transformation method greatly simplifies the the algebra especially when the external force term of the Hamiltonian has a complex form having mixed variables like in nonlinear focusing systems. It does not require one to Taylor expand the Hamiltonian and does not require one to perform a reverse transformation at every intermediate stage. Moreover, the reverse transformation needs be done only if required. In conclusion, we state that although the method was applied to an intense beam with linear focusing, its application is more general.

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