EQUILIBRIUM BEAM INVARIANTS OF AN ELECTRON STORAGE RING WITH LINEAR x-y COUPLING *

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Abstract

In accelerators, it is common that the motion of the horizontal x-plane is coupled to that of the vertical y-plane. Such coupling will induce tune shifts and can cause instabilities. The damping and diffusion rates are also affected, which in turn will lead to a change in the equilibrium invariants. With the perturbative approach which is also used for synchrobetatron coupling [B. Nash, J. Wu, and A. Chao, work in progress], we study the x-y coupled case in this paper. Starting from the one-turn map, we give explicit formulae for the tune shifts, damping and diffusion rates, and the equilibrium invariants. We focus on the cases where the system is near the integer or half integer, and sum or difference resonances where small coupling can cause a large change in the beam distribution.

Introduction

It is of general interest to obtain equilibrium invariants for a coupled system. In this paper, we will find the equilibrium value of the eigen-invariants for a linear x-y coupled system. Particularly, we study their behavior near resonances, *i.e.,* integer / half-integer, and sum / difference resonances. In general, for a 3-D system not exactly on resonance, there are three eigen-invariants $g_{1,2,3}$. Assuming no coupling between the longitudinal and the transverse dimensions, we can consider a 2-D system with the two transverse dimensions (even though the diffusion matrix has to be deduced from 3-D dynamics), and we will work in the betatron coordinates. It can be shown that $g_i = Z^T G_i Z/2$ are eigen-invariants with the matrix $G_i = JU\bar{G}_iU^TJ$, and $Z = [x_\beta, x'_\beta, y_\beta, y'_\beta]^T$. We describe the dynamics by a one-turn map, M. The eigenvector matrix U is defined by $MU = U\lambda$, with λ being the diagonal eigenvalue matrix.[1] Assuming the damping and diffusion are slow processes, and the particle motion still follows the eigen-invariants. The change in the invariant per turn is given by $\Delta\langle g_i\rangle = \oint ds(-\mathrm{tr}(A_i(s))\langle g_i\rangle + \mathrm{tr}(G_i(s)D(s)) \equiv -\oint b_i(s)\langle g_i\rangle + \oint d_i(s)$, which determines the equilibrium of $\langle g_i\rangle_{\mathrm{eq}} = \oint d_i(s)/\oint b_i(s)$. Here, $D(s)$ is the local diffusion matrix, and $A(s) = U^{-1}b(s)U$ with $b(s)$ the local damping matrix, A_1 the up-left 2×2 matrix and A_2 the low-right 2×2 matrix of A. So technically, the problem reduces to finding the U-matrix. This is accomplished via the 2nd-order degenerate perturbation theory.

Second-order degenerate perturbation theory

The uncoupled motion in the x and y dimension is described by the one-turn map $\begin{pmatrix} M_{x0} & 0 \\ 0 & M_{y0}\end{pmatrix}$, where M_{x0} and M_{y0} are symplectic 2×2 matrices. In the integer (half-integer) resonance case, $M_{y0} = I$ $(-I)$. Following Courant-Synder [1], we can write $M_{x0,y0} = \cos\mu_{x,y}\,I + \sin\mu_{x,y}\,J_{x,y}$ with $J_{x,y} = \begin{pmatrix} \alpha_{x,y} & \beta_{x,y} \\ -\gamma_{x,y} & -\alpha_{x,y}\end{pmatrix}$. The corresponding eigenvectors are $v_{10} = [(1-i\alpha_x)/\sqrt{\gamma_x}, i\sqrt{\gamma_x}, 0, 0]^T/\sqrt{2}$ for the positive mode, *i.e.,* its eigenvalue $\lambda_{10} = e^{i\mu_x}$. The negative mode $\lambda_{-10} = e^{-i\mu_x}$ has $v_{-10} = iv^*_{10}$. Same for the y dimension.

We solve the eigenequation

$$M v_k = \lambda_k v_k, \tag{1}$$

with $M = M_0 + M_1 + M_2$. The four eigenvectors v_{k0} ($k = 1, -1, 2,$ and -2) of M_0 set up the complete and orthonormal basis, with $M_0 v_{k0} = \lambda_{k0} v_{k0}$. The conjugate vector is defined as $v^{k0} \equiv -i\,\mathrm{sgn}(k) v^\dagger_{k0} J$, so that $v^{j0}v_{k0} = \delta_{jk}$. We will treat M_1 and M_2 as the 1st- and 2nd-order perturbation. The eigenvalues are expanded as $\lambda_k = \lambda_{k0} + \lambda_{k1} + \lambda_{k2} + \mathcal{O}(\epsilon^3)$. Assuming that there is degeneracy among vectors with indices $\in Z_{dg}$, the eigenvectors are expanded in the following way

$$v_k = \begin{cases} \left[1 + c^k_{k2} + \mathcal{O}(\epsilon^3)\right] v_{k0} \\ \quad + \sum_{j\neq k}\left[c^j_{k1} + c^j_{k2} + \mathcal{O}(\epsilon^3)\right] v_{j0} \\ \qquad \text{for} \quad k \notin Z_{dg} \\ \sum_{j\in Z_{dg}}\left[c^j_{k0} + c^j_{k2} + \mathcal{O}(\epsilon^3)\right] v_{j0} \\ \quad + \sum_{j\notin Z_{dg}}\left[c^j_{k1} + c^j_{k2} + \mathcal{O}(\epsilon^3)\right] v_{j0} \\ \qquad \text{for} \quad k \in Z_{dg}. \end{cases} \tag{2}$$

Now, the eigenequation (1) is solved order by order. Results are given below omitting derivations.

Nondegenerate part (*i.e.,* for $k \notin Z_{dg}$) For the 1st-order, we have $\lambda_{k1} = \mathcal{M}_{kk}$; and for $l \neq k$, $c^l_{k1} = \mathcal{M}_{lk}/(\lambda_{k0} - \lambda_{l0})$ with $\mathcal{M}_{lk} \equiv v^{l0}M_1 v_{k0}$. For the 2nd-order, we have $\lambda_{k2} = \sum_{j\neq k} c^j_{k1}\mathcal{M}_{kj} + \mathcal{M}_{2,kk}$; and for $l \neq k$, $c^l_{k2} = (c^l_{k1}\lambda_{k1} - \sum_{j\neq k} c^j_{k1}\mathcal{M}_{lj} - \mathcal{M}_{2,lk})/(\lambda_{l0} - \lambda_{k0})$ with $\mathcal{M}_{2,lk} \equiv v^{l0}M_2 v_{k0}$.

Degenerate part (*i.e.,* for $k \in Z_{dg}$) For the 1st-order, for $l \in Z_{dg}$, we have

$$\sum_{j\in Z_{dg}} \mathcal{M}_{lj} c^j_{k0} = \lambda_{k1} c^l_{k0}, \tag{3}$$

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[1] Various matrices are $J = \begin{pmatrix} J_2 & 0 \\ 0 & J_2\end{pmatrix}$, $\bar{G}_1 = \begin{pmatrix} i\bar{\sigma}_x & 0 \\ 0 & 0\end{pmatrix}$, $\bar{G}_2 = \begin{pmatrix} 0 & 0 \\ 0 & i\bar{\sigma}_x\end{pmatrix}$, $J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}$, $\bar{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}$. Superscript T is taking transpose, tr takes the trace.

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which is the eigenequation for both λ_{k1} and c^l_{k0}; and for $l \notin Z_{dg}$, $c^l_{k1} = (\sum_{j\in Z_{dg}} c^j_{k0}\mathcal{M}_{lj})/(\lambda_{k0} - \lambda_{l0})$. For the 2nd-order, for $l \in Z_{dg}$, we have

$$\lambda_{k2} = \frac{\sum_{j\notin Z_{dg}} c^j_{k1}\mathcal{M}_{lj} + \sum_{j\in Z_{dg}} c^j_{k0}\mathcal{M}_{2,lj}}{c^l_{k0}}; \quad (4)$$

and for $l \notin Z_{dg}$, $c^l_{k2} = (\lambda_{k1}c^l_{k1} - \sum_{j\notin Z_{dg}} c^j_{k1}\mathcal{M}_{lj} - \sum_{j\in Z_{dg}} c^j_{k0}\mathcal{M}_{2,lj})/(\lambda_{l0} - \lambda_{k0})$. Notice that the v_k in Eq. (2) are not normalized yet.

Resonances

In the integer / half-integer resonance case, the degeneracy comes in one subspace, say, in y, so that $\lambda_{20} = \lambda_{-20}$, and $Z_{dg} = (2,-2)$. For integer, then $\lambda_{20} = \lambda_{-20} \to 1$, for half-integer $\to -1$. For sum resonance, *i.e.*, $\mu_x + \mu_y \approx 2n\pi$, $Z_{dg} = (1,-2)$. For difference resonance, *i.e.*, $\mu_x - \mu_y \approx 2n\pi$, $Z_{dg} = (1,2)$. Hence, calculations for all cases involve eigenanalyzing 2×2 coupling coefficient matrices, which we designate as $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Its eigenvalues are $\lambda_\pm = \frac{1}{2}[(a+d) \pm \sqrt{(a-d)^2 + 4bc}]$ and eigenvectors are $v_\pm = (\frac{1}{2c}[(a-d) \pm \sqrt{(a-d)^2+4bc}], 1)^T$.

Sum resonance According to Eq. (3),

$$\begin{cases} \begin{pmatrix} \mathcal{M}_{11} & \mathcal{M}_{1-2} \\ \mathcal{M}_{-21} & \mathcal{M}_{-2-2} \end{pmatrix} \begin{pmatrix} c^1_{10} \\ c^{-2}_{10} \end{pmatrix} = \lambda_{11} \begin{pmatrix} c^1_{10} \\ c^{-2}_{10} \end{pmatrix}, \\ \begin{pmatrix} \mathcal{M}_{11} & \mathcal{M}_{1-2} \\ \mathcal{M}_{-21} & \mathcal{M}_{-2-2} \end{pmatrix} \begin{pmatrix} c^1_{-20} \\ c^{-2}_{-20} \end{pmatrix} = \lambda_{-21} \begin{pmatrix} c^1_{-20} \\ c^{-2}_{-20} \end{pmatrix}. \end{cases}$$

Explicitly, we have $a = \mathcal{M}_{11} = i\lambda_{10}\lfloor\mathcal{M}_{11}\rceil, d = \mathcal{M}_{-2-2} = -i\lambda^*_{20}\lfloor\mathcal{M}_{22}\rceil, b = \mathcal{M}_{1-2} \equiv (\xi/2)e^{i\phi}$, and $c = \mathcal{M}_{-21} \equiv (\xi/2)e^{i(2\mu-\phi)}$. Here, we introduce the operator $\lfloor\ \rceil$, which only means that $\lfloor x\rceil$ is real, but not guarantee that $\lfloor x\rceil > 0$. Notice that $\lambda_{10} = \lambda_{-20} = e^{i\mu}$. We define $\mathcal{M}_{11} - \mathcal{M}_{-2-2} = ie^{i\mu}(\lfloor\mathcal{M}_{11}\rceil + \lfloor\mathcal{M}_{22}\rceil) \equiv ie^{i\mu}\Delta\mu$. We then define $\tanh\theta \equiv \xi/|\Delta\mu|$. The eigenvectors depend on the sign of $\Delta\mu$. For $\Delta\mu > 0$,

$$\begin{cases} \begin{pmatrix} c^1_{10} \\ c^{-2}_{10} \end{pmatrix} = \begin{pmatrix} ie^{i(\phi-\mu)}\cosh(\theta/2) \\ \sinh(\theta/2) \end{pmatrix}, \\ \begin{pmatrix} c^1_{-20} \\ c^{-2}_{-20} \end{pmatrix} = \begin{pmatrix} \sinh(\theta/2) \\ -ie^{-i(\phi-\mu)}\cosh(\theta/2) \end{pmatrix}. \end{cases}$$

Notice that in this formalism, the system is unstable for the x-y coupled case.[2] Now the U-matrix is constructed as $U = (v_1, iv_1^*, v_2[= iv^*_{-2}], v_{-2})$, with

$$\begin{cases} v_1 = ie^{i(\phi-\mu)}\cosh(\theta/2)\, v_{10} + \sinh(\theta/2)\, v_{-20} \\ v_{-2} = \sinh(\theta/2)\, v_{10} - ie^{-i(\phi-\mu)}\cosh(\theta/2)\, v_{-20} \end{cases}.$$

In β-coordinates, the damping and diffusion matrices read

$$b = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2b_x & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2b_y \end{pmatrix}, \text{ and}$$

$$D = d\begin{pmatrix} \eta_x^2 & \eta_x\eta'_x & \eta_x\eta_y & \eta_x\eta'_y \\ \eta_x\eta'_x & \eta'^2_x & \eta'_x\eta_y & \eta'_x\eta'_y \\ \eta_x\eta_y & \eta'_x\eta_y & \eta_y^2 & \eta_y\eta'_y \\ \eta_x\eta'_y & \eta'_x\eta'_y & \eta_y\eta'_y & \eta'^2_y \end{pmatrix}$$

Using $A = U^{-1}bU$, the damping coefficients are given as

$$\begin{cases} b_1 = 2b_x\cosh^2(\theta/2) - 2b_y\sinh^2(\theta/2), \\ b_2 = -2b_x\sinh^2(\theta/2) + 2b_y\cosh^2(\theta/2). \end{cases},$$

and diffusion coefficients [3]

$$\begin{aligned} d_1 &= d\mathcal{H}_x\cosh^2(\theta/2) + d\mathcal{H}_y\sinh^2(\theta/2) \\ &- \frac{d\sinh(\theta)}{\sqrt{\gamma_x\gamma_y}}\left[\left(\mathcal{G}_x\mathcal{G}_y - \eta'_x\eta'_y\right)\cos(\mu-\phi)\right. \\ &+ \left.\left(\mathcal{G}_x\eta'_y + \mathcal{G}_y\eta'_x\right)\sin(\mu-\phi)\right], \end{aligned} \quad (5)$$

$$\begin{aligned} d_2 &= d\mathcal{H}_x\sinh^2(\theta/2) + d\mathcal{H}_y\cosh^2(\theta/2) \\ &- \frac{d\sinh(\theta)}{\sqrt{\gamma_x\gamma_y}}\left[\left(\mathcal{G}_x\mathcal{G}_y - \eta'_x\eta'_y\right)\cos(\mu-\phi)\right. \\ &+ \left.\left(\mathcal{G}_x\eta'_y + \mathcal{G}_y\eta'_x\right)\sin(\mu-\phi)\right], \end{aligned} \quad (6)$$

with $\mathcal{H}_{x,y} = \gamma_{x,y}\eta^2_{x,y} + 2\alpha_{x,y}\eta_{x,y}\eta'_{x,y} + \beta_{x,y}\eta'^2_{x,y}$, $\mathcal{G}_{x,y} = \alpha_{x,y}\eta'_{x,y} + \gamma_{x,y}\eta_{x,y}$, and $\gamma_{x,y}\mathcal{H}_{x,y} = \eta'^2_{x,y} + \mathcal{G}^2_{x,y}$.

The equilibrium value is $\langle g_i\rangle_{\rm eq} = \oint d_i(s)/\oint b_i(s)$, for $i = 1,2$.

Difference resonance According to Eq. (3)

$$\begin{cases} \begin{pmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} \\ \mathcal{M}_{21} & \mathcal{M}_{22} \end{pmatrix} \begin{pmatrix} c^1_{10} \\ c^2_{10} \end{pmatrix} = \lambda_{11} \begin{pmatrix} c^1_{10} \\ c^2_{10} \end{pmatrix}, \\ \begin{pmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} \\ \mathcal{M}_{21} & \mathcal{M}_{22} \end{pmatrix} \begin{pmatrix} c^1_{20} \\ c^2_{20} \end{pmatrix} = \lambda_{21} \begin{pmatrix} c^1_{20} \\ c^2_{20} \end{pmatrix}. \end{cases}$$

Explicitly, we have $a = \mathcal{M}_{11} = i\lambda_{10}\lfloor\mathcal{M}_{11}\rceil$, $d = \mathcal{M}_{22} = i\lambda_{20}\lfloor\mathcal{M}_{22}\rceil$, $b = \mathcal{M}_{12} \equiv (\xi/2)e^{i\phi}$, and $c = \mathcal{M}_{21} = -(\xi/2)e^{i(2\mu-\phi)}$. Notice that $\lambda_{10} = \lambda_{20} = e^{i\mu}$. Let us now define $\mathcal{M}_{11} - \mathcal{M}_{22} = ie^{i\mu}(\lfloor\mathcal{M}_{11}\rceil - \lfloor\mathcal{M}_{22}\rceil) \equiv ie^{i\mu}\Delta\mu$. We then define $\tan\theta \equiv \xi/|\Delta\mu|$. Again, the eigenvectors depend on the sign of $\Delta\mu$. For $\Delta\mu > 0$, they are

$$\begin{cases} \begin{pmatrix} c^1_{10} \\ c^2_{10} \end{pmatrix} = \begin{pmatrix} -ie^{i(\phi-\mu)}\cos(\theta/2) \\ \sin(\theta/2) \end{pmatrix}, \\ \begin{pmatrix} c^1_{20} \\ c^2_{20} \end{pmatrix} = \begin{pmatrix} \sin(\theta/2) \\ -ie^{-i(\phi-\mu)}\cos(\theta/2) \end{pmatrix}. \end{cases}$$

Notice that the system is found to be stable for the x-y coupled case.[4] Now the U-matrix is constructed as $U = (v_1, iv_1^*, v_2, iv_2^*)$, with

$$\begin{cases} v_1 = -ie^{i(\phi-\mu)}\cos(\theta/2)\, v_{10} + \sin(\theta/2)\, v_{20} \\ v_2 = \sin(\theta/2)\, v_{10} - ie^{-i(\phi-\mu)}\cos(\theta/2)\, v_{20} \end{cases}.$$

The damping coefficients are computed to be

$$\begin{cases} b_1 = 2b_x\cos^2(\theta/2) + 2b_y\sin^2(\theta/2), \\ b_2 = 2b_x\sin^2(\theta/2) + 2b_y\cos^2(\theta/2). \end{cases}$$

and diffusion coefficients are

$$d_1 = d\mathcal{H}_x\cos^2(\theta/2) + d\mathcal{H}_y\sin^2(\theta/2)$$

[2] For $\Delta\mu < 0$, we have $\begin{cases} \begin{pmatrix} c^1_{10} \\ c^{-2}_{10} \end{pmatrix} = \begin{pmatrix} -ie^{i(\phi-\mu)}\sinh(\theta/2) \\ \cosh(\theta/2) \end{pmatrix}, \\ \begin{pmatrix} c^1_{-20} \\ c^{-2}_{-20} \end{pmatrix} = \begin{pmatrix} \cosh(\theta/2) \\ ie^{-i(\phi-\mu)}\sinh(\theta/2) \end{pmatrix}. \end{cases}$

[3] Notice that for $\eta'_{x,y} = 0$, $d_1 = d\mathcal{H}_x\cosh^2(\theta/2) + d\mathcal{H}_y\sinh^2(\theta/2) - d\sqrt{\mathcal{H}_x\mathcal{H}_y}\cos(\mu-\phi)\sinh(\theta)$, and $d_2 = d\mathcal{H}_x\sinh^2(\theta/2) + d\mathcal{H}_y\cosh^2(\theta/2) - d\sqrt{\mathcal{H}_x\mathcal{H}_y}\cos(\mu-\phi)\sinh(\theta)$.

[4] For $\Delta\mu < 0$, they are $\begin{cases} \begin{pmatrix} c^1_{10} \\ c^2_{10} \end{pmatrix} = \begin{pmatrix} -ie^{i(\phi-\mu)}\sin(\theta/2) \\ \cos(\theta/2) \end{pmatrix}, \\ \begin{pmatrix} c^1_{20} \\ c^2_{20} \end{pmatrix} = \begin{pmatrix} \cos(\theta/2) \\ -ie^{-i(\phi-\mu)}\sin(\theta/2) \end{pmatrix}. \end{cases}$

$$
\begin{aligned}
&- \frac{d\sin(\theta)}{\sqrt{\gamma_x\gamma_y}}\left[\left(\mathcal{G}_x\mathcal{G}_y+\eta_x'\eta_y'\right)\sin(\mu-\phi)\right.\\
&+ \left.\left(\mathcal{G}_x\eta_y'-\mathcal{G}_y\eta_x'\right)\cos(\mu-\phi)\right], \qquad (7)
\end{aligned}
$$

$$
\begin{aligned}
d_2 &= d\mathcal{H}_x\sin^2(\theta/2)+d\mathcal{H}_y\cos^2(\theta/2)\\
&+ \frac{d\sin(\theta)}{\sqrt{\gamma_x\gamma_y}}\left[\left(\mathcal{G}_x\mathcal{G}_y+\eta_x'\eta_y'\right)\sin(\mu-\phi)\right.\\
&+ \left.\left(\mathcal{G}_x\eta_y'-\mathcal{G}_y\eta_x'\right)\cos(\mu-\phi)\right]. \qquad (8)
\end{aligned}
$$

Integer / half-integer resonance The physics of the sum / difference resonance is analyzed within the 1st-order degenerate perturbation as above. However, the integer / half-integer resonance is more involved. The physics of this resonance needs to be analyzed in a 2nd-order perturbation calculation. As we find, the perturbation matrices are $M_1 = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$, and $M_2 = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$. Equation (3) yields $\lambda_{k1} = 0$ for $k \in Z_{dg}$, so it is determined by Eq. (4), which is 2nd-order, *i.e.*,

$$
\left\{
\begin{aligned}
&\begin{pmatrix}
\frac{\mathcal{M}_{12}\mathcal{M}_{21}}{\lambda_{20}-\lambda_{10}} + \frac{\mathcal{M}_{-12}\mathcal{M}_{2-1}}{\lambda_{20}-\lambda_{-10}} + \mathcal{M}_{2,22} &
\frac{\mathcal{M}_{1-2}\mathcal{M}_{21}}{\lambda_{20}-\lambda_{10}} + \frac{\mathcal{M}_{-1-2}\mathcal{M}_{2-1}}{\lambda_{20}-\lambda_{-10}} + \mathcal{M}_{2,2-2} \\
\frac{\mathcal{M}_{12}\mathcal{M}_{-21}}{\lambda_{20}-\lambda_{10}} + \frac{\mathcal{M}_{-12}\mathcal{M}_{-2-1}}{\lambda_{20}-\lambda_{-10}} + \mathcal{M}_{2,-22} &
\frac{\mathcal{M}_{1-2}\mathcal{M}_{-21}}{\lambda_{20}-\lambda_{10}} + \frac{\mathcal{M}_{-1-2}\mathcal{M}_{-2-1}}{\lambda_{20}-\lambda_{-10}} + \mathcal{M}_{2,-2-2}
\end{pmatrix}
\begin{pmatrix} c_{20}^{2} \\ c_{20}^{-2} \end{pmatrix}
= \lambda_{22}\begin{pmatrix} c_{20}^{2} \\ c_{20}^{-2} \end{pmatrix}, \\
&\begin{pmatrix}
\frac{\mathcal{M}_{12}\mathcal{M}_{21}}{\lambda_{-20}-\lambda_{10}} + \frac{\mathcal{M}_{-12}\mathcal{M}_{2-1}}{\lambda_{-20}-\lambda_{-10}} + \mathcal{M}_{2,22} &
\frac{\mathcal{M}_{1-2}\mathcal{M}_{21}}{\lambda_{-20}-\lambda_{10}} + \frac{\mathcal{M}_{-1-2}\mathcal{M}_{2-1}}{\lambda_{-20}-\lambda_{-10}} + \mathcal{M}_{2,2-2} \\
\frac{\mathcal{M}_{12}\mathcal{M}_{-21}}{\lambda_{-20}-\lambda_{10}} + \frac{\mathcal{M}_{-12}\mathcal{M}_{-2-1}}{\lambda_{-20}-\lambda_{-10}} + \mathcal{M}_{2,-22} &
\frac{\mathcal{M}_{1-2}\mathcal{M}_{-21}}{\lambda_{-20}-\lambda_{10}} + \frac{\mathcal{M}_{-1-2}\mathcal{M}_{-2-1}}{\lambda_{-20}-\lambda_{-10}} + \mathcal{M}_{2,-2-2}
\end{pmatrix}
\begin{pmatrix} c_{-20}^{2} \\ c_{-20}^{-2} \end{pmatrix}
= \lambda_{-22}\begin{pmatrix} c_{-20}^{2} \\ c_{-20}^{-2} \end{pmatrix}.
\end{aligned}
\right.
$$

Notice that $d = a^*$ and $c = b^*$. We define $a - d = i2\Im a \equiv i\Delta\mu$, and $b = c^* \equiv (\xi/2)e^{i\phi}$. Notice that $\xi > 0$, however, $\Delta\mu$ can be either negative or positive. We then define $\tanh(\theta) \equiv \xi/|\Delta\mu|$. The eigenvectors depend on the sign of $\Delta\mu$.[5] For $\Delta\mu > 0$, they are $\left\{ \begin{array}{l} \begin{pmatrix} c_{20}^{2} \\ c_{20}^{-2} \end{pmatrix} = \begin{pmatrix} ie^{i\phi}\cosh(\theta/2) \\ \sinh(\theta/2) \end{pmatrix} \\ \begin{pmatrix} c_{-20}^{2} \\ c_{-20}^{-2} \end{pmatrix} = \begin{pmatrix} \sinh(\theta/2) \\ -ie^{-i\phi}\cosh(\theta/2) \end{pmatrix} \end{array} \right.$. The simplectic U-matrix is then defined as $U = \{v_1, iv_1^*, v_2, iv_2^*\}$, with $\left\{ \begin{array}{l} v_2 = ie^{i\phi}\cosh(\theta/2)\,v_{20} + \sinh(\theta/2)\,v_{-20} \\ v_{-2} = \sinh(\theta/2)\,v_{20} - ie^{-i\phi}\cosh(\theta/2)\,v_{-20} \end{array} \right.$.

[5] For $\Delta\mu < 0$, $\left\{ \begin{array}{l} \begin{pmatrix} c_{20}^{2} \\ c_{20}^{-2} \end{pmatrix} = \begin{pmatrix} -ie^{i\phi}\sinh(\theta/2) \\ \cosh(\theta/2) \end{pmatrix}, \\ \begin{pmatrix} c_{-20}^{2} \\ c_{-20}^{-2} \end{pmatrix} = \begin{pmatrix} \cosh(\theta/2) \\ ie^{-i\phi}\sinh(\theta/2) \end{pmatrix}. \end{array} \right.$

The damping coefficients are computed to be $\left\{ \begin{array}{l} b_1 = \mathrm{tr}_x(A) = A_{11} + A_{22} \approx 2b_x \\ b_2 = \mathrm{tr}_y(A) = A_{33} + A_{44} \approx 2b_y \end{array} \right.$. Similarly the diffusion coefficients are $d_1 = d\mathcal{H}_x$; and for $\Delta\mu > 0$,[6] $d_2 = d\mathcal{H}_y\cosh(\theta) + (d/\gamma_y)[\cos(\phi)(\eta_y'^2 - \mathcal{G}_y^2) + 2\sin(\phi)\eta_y'\mathcal{G}_y]\sinh(\theta)$.

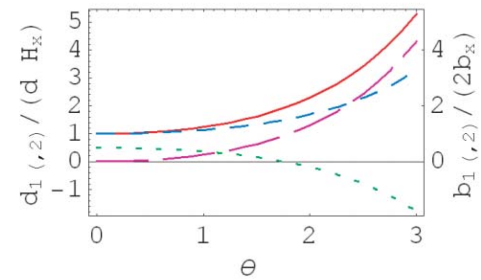

Figure 1: Damping / diffusion rate for the sum resonance.

Discussion

For integer / half-integer resonance, the coupling does not affect the damping. However, the diffusion rate may increase substantially, *i.e.*, $\Delta\mu \to \xi$ implies $\theta \gg 1$. In the sum resonance case, the instability comes from two effects. In addition to a coupling stopband, the damping rate may become negative (antidamping) while the diffusion rate becomes very large. In the difference resonance case, both the damping rate and the diffusion rate stay finite. Let us study the sum resonance, and show some properties in Fig. 1. We plot the simplified expression with $\eta_x' = \eta_y' = 0$ as in Footnote 3. Thinking of a flat beam, we assume $\mathcal{H}_y = \mathcal{H}_x/100$, also an exaggerated $b_y = b_x/2$, with parameters $\phi = \pi/2$ and $\mu = 0.3$. The red solid curve is for $d_1(\theta)$, the purple long-dashed for $d_2(\theta)$, the blue dashed for $b_1(\theta)$, and green dotted for $b_2(\theta)$. It is clearly seen that due to coupling (θ), the diffusion rate in y direction increases very rapidly. Interestingly, the y damping rate becomes negative, indicating an antidamping type of instability. Of course, this does not happen for $b_x = b_y$.

In conclusion, we studied the equilibrium value of the eigen-invariants near the integer / half-integer resonance, and the sum / difference resonance. The similar topic of synchro-betatron coupling is studied elsewhere [2], where more details can be found.

REFERENCES

[1] E.D. Courant and H.S. Synder, Annals of Phys. **3**, 1 (1958).

[2] B.E. Nash, J. Wu, and A.W. Chao, (work in progress, 2005).

[6] For $\Delta\mu < 0$, we have $d_2 = d\mathcal{H}_y\cosh\theta - (d/\gamma_y)[\cos\phi(\eta_y'^2 - \mathcal{G}_y^2) + 2\sin\phi\,\eta_y'\mathcal{G}_y]\sinh\theta$. Notice that the difference comes from the sign of $\Delta\mu$. This can be absorbed into the definition of θ.