TRANSVERSE IMPEDANCE OF ELLIPTICAL TAPERS*

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Abstract

For a slowly tapered vacuum chamber with confocal elliptical cross-section, results obtained with the ABCI and GdfidL electromagnetic simulators are compared with analytic estimates for the transverse impedance at zero frequency. In the limits of round and flat chambers, we discuss the conditions for validity of the analytic approximations of Yokoya and Stupakov.

INTRODUCTION

An important issue in the design of modern synchrotron light sources is the determination of the transverse impedance of vacuum chambers for small-gap undulators which have tapers going from one cross-section to another in a given length. In the case of interest, the tapering is sufficiently gradual to be effective, the chamber height changes by a substantial factor, and the horizontal-to-vertical aspect ratio is large.

Yokoya has derived the low frequency transverse impedance of an axially symmetric tapered transition [1],

$$Z_{\perp}^{round}(k) \cong -\frac{iZ_0}{2\pi}\int_{-\infty}^{\infty} dz \frac{r'(z)^2}{r(z)^2}, \qquad (1)$$

where k is the wavenumber of the perturbing field, Z_0 the free space impedance, $r(z)$ the radius of the tapered circular chamber, and prime denotes derivative with respect to z. When the variation of the radius takes place over a distance ℓ, Stupakov [2] has presented arguments indicating that Yokoya's approximation should be valid when

$$kr^2/\ell << 1. \qquad (2)$$

In addition, he used first-order perturbation theory to provide a new derivation of Yokoya's result for $k=0$, based on the solution of electrostatic and magnetostatic problems. We have extended Stupakov's calculation to higher-order [3] and have found that the Yokoya approximation Eq. (1) is the first term in an asymptotic expansion in the parameter r_{av}^2/ℓ^2, where r_{av} stands for the average value of *r(z)*. The higher-order terms are found to reduce the impedance below the value found by Yokoya. This has also been observed in numerical calculations using ABCI [5] as illustrated in Fig. 1.

Using electrostatic and magnetostatic calculations, Stupakov [4] has also obtained a result within first-order perturbation theory for the impedance at k=0 of a flat rectangular chamber of constant half-width w and varying half-height $h(z)$, in the case when the width is large compared with the height. His result is

$$Z_y^{flat}(0) = -\frac{iZ_0 w}{2}\int_{-\infty}^{\infty} dz \frac{h'(z)^2}{h(z)^3}. \qquad (3)$$

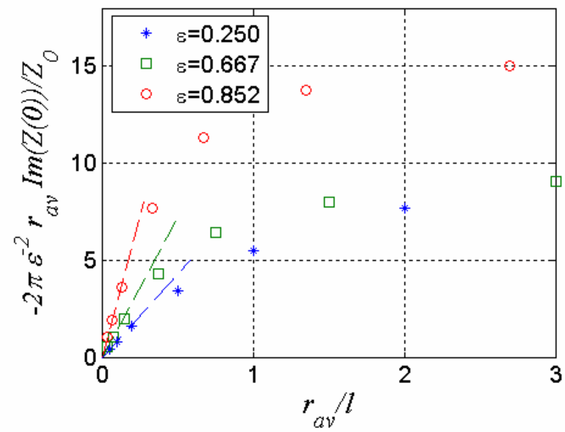

Figure 1: Impedance of a linearly tapered symmetric collimator for different $\varepsilon=(r_{max}-r_{min})/(r_{max}+r_{min})$. Broken lines show Yokoya approximation Eq. (1).

This expression has a form similar to Eq. (1) for a circular pipe, but is larger by a factor $\sim \pi w/h$. Applying an argument derived from considerations equating the phase velocity of the slowest eigenmode of the wave guide formed by the beam pipe to the phase velocity of the beam–induced image current [2], Stupakov suggests that Eq. (3) should be a good approximation for

$$kw^2h'/h << 1. \qquad (4)$$

At $k=0$, this condition does not provide a restriction on the width w, so the vertical impedance increases indefinitely as the chamber widens. However, we found [3] that Eq. (3) is the first term in an expansion in the parameter $h_{av}w/\ell^2$, where h_{av} is the average chamber height. For large w higher-order terms reduce $\operatorname{Im} Z_y^{flat}(0)$ below the value given in Eq. (3) and the growth saturates.

In order to investigate this further we chose to analyze the problem of an elliptical chamber with varying ellipticity. This allowed us to smoothly extrapolate between the cases of circular and flat chambers.

ELLIPTICAL CHAMBER

Stupakov's [2] method involves solving inhomogeneous two-dimensional Poison equations. Matching boundary conditions for a translation invariant (uniform) elliptical cross-section is most easily done using elliptic cylindrical coordinates (μ,θ,z). The contour surfaces of constant μ are confocal elliptical cylinders, while those of constant θ are confocal hyperbolic cylinders. The confocal cylinder $\mu=\rho$ forms the inner beam pipe boundary, while the *z*-axis is directed along the chamber axis. The relationship between Cartesian and elliptic coordinates is

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given by

$$x = A\cosh\mu\cos\theta$$
$$y = A\sinh\mu\sin\theta \qquad (5)$$
$$z = z$$

where *2A* is the distance between the foci. The major and minor semi-axes of the elliptical cross-section are $a = A\cosh\rho$ and $b = A\sinh\rho$, and its eccentricity is given by $e = \sqrt{1-b^2/a^2} = 1/\cosh\rho$. The limiting case of a circular cross-section of radius r is given by $A = 2r\exp(-\rho)$, with $\rho\to\infty$. The limit of $\rho\to 0$, approximates a flat pipe $2h = 2A\rho$ high by $2w = 2A$ wide.

In the case when the elliptical cross-section varies with *z*, it is generally not possible to introduce an orthogonal coordinate system that matches the cross-section at each value of *z*. However, when the variation maintains a confocal structure, we can use the elliptic cylindrical coordinates introduced above, allowing ρ to have a *z*-dependence. The aspect ratio is given by $b(z)/a(z) = \tanh\rho(z)$ and the constant A is determined by $A^2 = a(z)^2 - b(z)^2$. The requirement for confocal variation allows arbitrary variation in one plane, e.g. arbitrary beam pipe half-height $b(z)$. The variation in the other plane is then fixed as soon as a is specified at any single value of *z*.

Within first-order perturbation theory [3], we find the vertical impedance is given by

$$Z_y(0) = \frac{iZ_0}{4\pi}\sum_{\substack{n=1\\ odd}}^{\infty}\int_{-\infty}^{\infty} dz\,\rho'(z)\frac{I_n^y + I_{n+2}^y}{2(n+1)}F_n^y \qquad (6)$$

where

$$F_n^y = \frac{n\cosh(n+2)\rho}{\sinh n\rho} + \frac{(n+2)\cosh n\rho}{\sinh(n+2)\rho} + \frac{n\sinh(n+2)\rho\cosh n\rho}{\sinh^2 n\rho} + \frac{(n+2)\sinh n\rho\cosh(n+2)\rho}{\sinh^2(n+2)\rho} \qquad (7)$$

and

$$I_n^y = \frac{\partial}{\partial z}\frac{e^{-n\rho(z)}}{\sinh n\rho(z)} = -\frac{n\rho'(z)}{\sinh^2 n\rho(z)}. \qquad (8)$$

The expression for the horizontal impedance $Z_x(0)$ is obtained by making the interchange $\cosh\leftrightarrow\sinh$ in Eqs. (7) and (8).

For a weakly elliptical pipe $(\rho >> 1)$, each term in the sum in Eq. (6) is $O(e^{-2\rho})$ smaller than its predecessor. Keeping only the lowest order correction, we get

$$Z_{y,x}(0) = -\frac{iZ_0}{2\pi}\int_{-\infty}^{\infty}\rho'(z)^2\left[1 \pm 4e^{-2\rho(z)}\right]. \qquad (9)$$

The plus sign is for the vertical impedance and the minus sign for the horizontal. For a circular pipe, $\rho'(z) = r'(z)/r(z)$, hence the dominant term is Eq. (1) originally found by Yokoya for a round tapered pipe.

In the limit of a flat pipe $(\rho\to 0)$, the sum in Eq. (6) can be replaced by an integral and to leading order

$$Z_y(0) \cong -\frac{iZ_0}{4\pi}\int_{-\infty}^{\infty} dz\frac{\rho'(z)^2}{\rho(z)^3}, \qquad (10)$$

and

$$Z_x(0) \cong -\frac{iZ_0}{4\pi}\int_{-\infty}^{\infty} dz\frac{\rho'(z)^2}{\rho(z)^2}. \qquad (11)$$

For an almost flat elliptical pipe, the width is approximately constant, and the half-height is $A\rho(z)$. Eq. (10) for the vertical impedance differs from the result of Stupakov Eq. (3) for a flat rectangular pipe by a factor of $(2\pi)^{-1}$. Eq. (11) for the horizontal impedance is one-half the Yokoya result of Eq. (1) for an axially symmetric tapered transition with matching vertical dimensions.

In the case of large ρ, corresponding to an almost round chamber, the first-order perturbation theory results are accurate for slowly tapered structures with r_{av}^2/ℓ^2 small. For small ρ, corresponding to large horizontal to vertical aspect ratio, the first-order perturbation result is a good approximation [3] only when the expansion parameter $A^2\rho_{av}/\ell^2 \cong wh_{av}/\ell^2$ is small. Otherwise, higher-order terms in the perturbation expansion cannot be neglected and first-order perturbation theory gives an over-estimate of the impedance. The breakdown of first-order perturbation theory is clearly demonstrated by Eq. (11). It predicts a substantial value for the horizontal impedance, but it is intuitively obvious that beam in a flat chamber should not experience any horizontal force.

GDFIDL CALCULATIONS

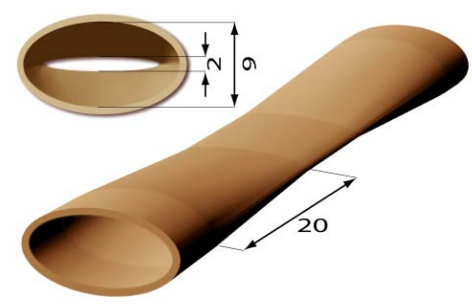

Figure 2: Geometry with principal dimensions in cm.

We now discuss calculations performed using code GdfidL [6] for the geometry of Fig. 2, consisting of a uniform pipe with elliptical cross-section linearly tapered to another uniform elliptical pipe with a smaller cross-section confocal to the first pipe; the structure is then continued mirror symmetrically with respect to the middle of the center pipe. Strictly speaking the linear tapering we use violates the confocal condition $a^2 - b^2 = const$ at intermediate cross-sections, however, for gradual transitions of interest the deviation is found to be quite small.

The vertical and axial dimensions are fixed for all calculations described below. Both the outer and the inner pipes are then varied from round to approximately flat maintaining the confocal condition. In each case, we separately calculate the horizontal and vertical wake-potentials by displacing the drive bunch in the appropriate plane. GdfidL calculates the EM fields and integrates them along the trajectory of the drive bunch. We take advantage of the symmetry planes and perform the calculations for a quarter structure, x>0, y>0, enforcing electric boundary condition in the y=0 (x=0) plane and magnetic boundary condition in x=0 (y=0) plane when calculating vertical (horizontal) wake-potentials.

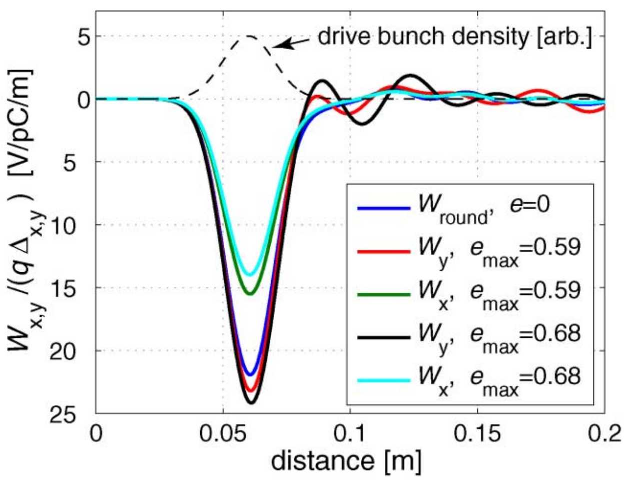

Figure 3: GdfidL wake-potentials.

A sample of GdfidL results is shown in Fig 3. The wake-potential due to a Gaussian bunch has a part that is approximately Gaussian, corresponding to the Yokoya contribution, and there is some additional ringing at the tail. As we go from round to a more flat geometry, there is an increase (decrease) in the vertical (horizontal) wake potential and an increase in the ringing. Note that we characterize the degree of flatness for a given structure in terms of the eccentricity e_{max} of the ellipse that makes the middle pipe cross-section.

The zero-frequency transverse (m=1) impedance is related to the wake-potential of a finite length bunch of charge q offset by Δ via $Z_{x,y}(0)=\frac{i}{qc\Delta_{x,y}}\int_{-\infty}^{\infty}dz\,W_{x,y}(z)$. In order to carefully account for the contribution of the ringing part, one must calculate the wake-potentials up to large distances behind the bunch. This is difficult and complicated by the fact that (at least for certain combinations of bunch lengths and mesh sizes) we have observed some systematic errors in the long-range wake-potentials, e.g. a non-vanishing DC value. Since narrow band resonances are not the focus of this paper, we have chosen to approximately calculate the zero frequency impedance from the front part of the wake-potential by $Z_{x,y}(0)=\frac{2i}{qc\Delta_{x,y}}\int_{-5\sigma_z}^{z_{@W\max}}dz\,W_{x,y}(z)$, where $z_{@W\max}$ stands for the location of the maximum (absolute) value of the wake-potential.

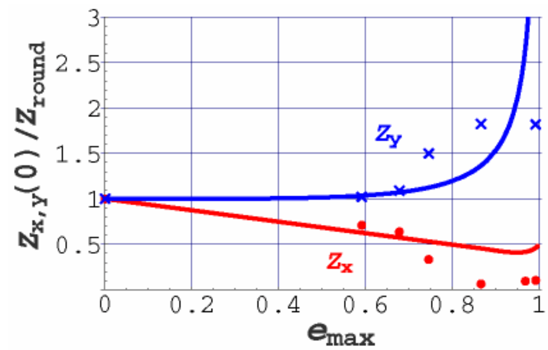

Figure 4: Transverse impedances from GdfidL calculations (symbols) and from Eqs. 6-8 (solid) as a function of ellipse eccentricity in the middle pipe.

Transverse impedances found in this manner are shown in Fig. 4. The impedance is normalized by its (GdfidL) value for axially symmetric pipe with the same vertical dimensions. For the 0.4 mm step size used the GdfidL value for the round pipe (~1.8 kΩ/m) is quite close to 1.7 kΩ/m that follows from Eq. (1). For small eccentricity the horizontal and vertical impedances follow Eq. (9). For larger e the vertical impedance plateaus at approximately twice the "round pipe" value of Eq. (1), and the horizontal impedance at less than 20% of this value. Hence, for large aspect ratios the impedance deviates from the asymptotic behaviour predicted by the first-order perturbation theory results of Eqs. (10-11), saturating at much smaller values.

While there is still some uncertainty in our present GdfidL calculations, and we are in the process of refining them, we believe the present results convincingly illustrate the breakdown of the first-order perturbation theory when the eccentricity is large. Finally, we note that similar observations of saturating vertical impedance and small horizontal impedance for large aspect ratio tapered beam pipe (although not in confocal elliptical geometry) have been reported by others [7-8].

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