ANALYTIC DESCRIPTION OF THE PHASE SLIP EFFECT IN RACE-TRACK MICROTRONS*

Yu.A. Kubyshin#, J. P. Rigla, Technical Univ. of Catalonia, Barcelona, Spain
A.V. Poseryaev, V.I. Shvedunov, SINP, Moscow State University, Russia

Abstract

Implementation of low energy injection schemes in the race-track microtron (RTM) design requires a better understanding of the longitudinal beam dynamics. Differently to the high energy case a low-energy beam will slip in phase relative to the accelerating structure phase. This phase slip is due to various features of the beam dynamics, in particular it is caused by the fact that at the first orbits the relativistic factor of the electrons $\beta < 1$. A phase shift may also be caused by the deformation of the particle trajectories due to the complicated magnetic field profile or simply by the fringe field in the end magnets. In article [1] we proposed the concept of equilibrium or synchronous particle for the case of non-relativistic energies in RTMs. An analytic approach for the description of the synchronous phase slip was developed and explicit, though approximate, formulas which allow to define the equilibrium injection phase and to fix the parameters of the accelerator were derived. In the present article we generalize this formalism for the case of non-trivial profile of the magnetic field in the RTM end magnets. Analytic formulas are derived, their accuracy can be improved by applying certain algorithm. Two examples of application of this formalism for fixing the injection phase and RTM parameters are given.

INTRODUCTION

It is well known that race-track microtrons, combining properties of a linear accelerator and a circular machine, are optimal as a source of electron beam for applications which require modest beam current and relatively high beam energy [2]. Because of the large energy gain per turn and the phase slip both in the drift space between the end magnets and in their fringe field, the analysis of the longitudinal dynamics in RTMs turns out to be quite complicated. Though there exist codes (RTMTRACE [3] and others) which permit to simulate the beam dynamics with sufficient precision, little analytic studies have been done so far. However, when designing a new accelerator with beam parameters quite different from those of known RTMs it is important to have a reliable model of longitudinal motion in order to gain a good understanding of the machine behaviour and choose and optimize its parameters. In article [1] we studied the phase slip effect and introduced a concept of synchronous particle with a relativistic factor $\beta < 1$ for the RTM model without fringe field in the end magnets. Here we generalize this result to the case of an arbitrary profile of the magnetic field at the end magnet entrance. Some numerical examples illustrating how our approach can be used for the accelerator design are presented.

*Work supported by grants PCI2005-A7-0284 and FIS2006-07016 of the Spanish Ministry of Science and Education
#iouri.koubychine@upc.edu

LONGITUDINAL DYNAMICS

Let us consider an electron RTM with the vertical component of the magnetic field induction in the end magnets described by a function $B(z)$. Without lost of generality we suppose that the field is absent, $B = 0$, for $z<0$ and that the z-axis is directed inside the magnet. An example of the function $B(z)$ of a magnetic system with inverse and main poles with fringe field included is shown in Fig. 1. For the sake of simplicity we will assume that the main pole has a region of constant magnetic field, i.e. there exists some $z_0 > 0$ such that $B(z) = B_0 = const$ for $z > z_0$. We also introduce the transversal horizontal component of the vector potential $A(z)$ such that $B(z) = dA/dz$

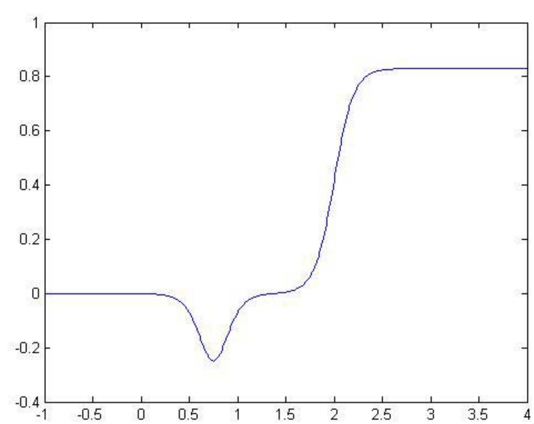

Figure 1. Magnetic field as a function of z in a magnetic system with inverse pole.

Let l be the straight section length, i.e. the separation between the end magnet entrances which coincide with the point z=0 (see Fig. 1) in the local coordinate system of each magnet. We suppose that the maximum energy gain in the linear accelerating structure (AS) is $\Delta E_{\max}$ and that the AS is modelled by an infinitely thin accelerating gap. As usual, the longitudinal dynamics of an individual particle is described by its energy E and phase φ with respect to the accelerating voltage. Let (φ_n, E_n) be the variables at the nth turn at the entrance of the AS. By φ_0 and E_0 we denote the phase and energy at the beginning of acceleration. We would like to note that in most of pulsed RTM designs the electrons after the injection and first passage through the AS are reflected back by the end magnet fringe field and/or an additional dipole. In this case E_0 is not the energy of injection but the energy before the second passage through the AS.

Let us recall that an RTM is designed in such a way that the so called equilibrium or synchronous particle moving with the velocity $v = c$ satisfies the condition of resonance motion:

$$T_{ns} = T_{RF}\left(\mu + \nu(n-1)\right), \tag{1}$$

i.e. the time of the nth revolution T_{ns} of such particle must be a multiple of the period of the RF field T_{RF}, where μ and ν are positive integers defining the mode of operation of the machine [2]. We will call such particle *ultra-relativistic,* or *asymptotic, synchronous* particle. Its longitudinal dynamics is characterized by a synchronous phase φ_s, so that its energy gain per turn is equal to $\Delta E_s = \Delta E_{\max}\cos\varphi_s$. The energy $E_{n,s}$ and phase $\varphi_{n,s}$ of the equilibrium particle at the nth turn are given by the following relations [1]:

$$\begin{aligned} E_{n,s} &= E_{0,s} + n\Delta E_s \\ \varphi_{n,s} &= \varphi_s + 2\pi n\left[\mu + \nu(n-1)/2\right] \end{aligned} \tag{2}$$

Let us consider the more common case when at the beginning of the acceleration the beam has $\beta = v/c < 1$. The general expression for the time of the nth revolution of a particle with energy E_n is given by

$$T_n = \frac{2l}{\beta(E_n)c} + \frac{2}{\beta(E_n)c}L(E_n), \tag{3}$$

where $\beta(E)$ is the relativistic factor β understood as a function of energy E and $L(E)$ is the length of the electron trajectory in the end magnet. To calculate this latter factor we introduce first the maximum penetration $z_{\max}(E)$ of the particle in the end magnet which is defined from the condition $|e|A(z_{\max}) = |p_z(E)|$, where p_z is the longitudinal particle momentum at the entrance of the magnet. Calculating the path length of a charged particle moving in a magnetic field characterized by the potential $A(z)$ one gets

$$L(E) = \int_0^{z_{\max}(E)} \frac{dz}{\sqrt{1 - \left(\dfrac{A(z)}{B_0 R(E)}\right)^2}}, \tag{4}$$

where

$$R(E) = \frac{E\beta(E)}{ecB_0} = \frac{\lambda\nu}{2\pi}\frac{E}{\Delta E_s}\beta(E) \tag{5}$$

is the orbit radius of an electron of energy E inside the region of constant magnetic field $B = B_0$ and λ is the RF field wavelength. It is clear that T_n cannot satisfy resonance condition (1) with integer μ and ν for all n. Nevertheless, there exists a phase space trajectory that approaches the ultra-relativistic synchronous one asymptotically.

The recursion relations between (φ_n, E_n) and (φ_{n+1}, E_{n+1}) are given by

$$\begin{aligned} E_{n+1} &= E_n + \Delta E_{\max}\cos\varphi_n \\ \varphi_{n+1} &= \varphi_n + K(E_{n+1}) \end{aligned} \tag{6}$$

with the phase advance given by the function

$$K(E) = \frac{4\pi l}{\lambda\beta(E)} + \frac{8\pi}{\lambda\beta(E)}L(E). \tag{7}$$

Of course, $K(E_n) = 2\pi T_n / T_{RF}$ with T_n given by Eq. (3). As the energy grows, the longitudinal phase coordinates get closer to those of the asymptotic synchronous particle, therefore it is reasonable to characterize (φ_n, E_n) with respect to the asymptotic synchronous trajectory (φ_{ns}, E_{ns}) described by formulas (2). To calculate the difference between these two trajectories we introduce the dimensionless parameter $\varepsilon_n = \Delta E_s / E_{n,s}$ which decreases with the growth of n. Expanding function (7) in powers of ε_n and applying the technique developed in Ref. [1] we obtain the following leading approximation to the solution of system of difference equations (6) without synchrotron oscillations:

$$\varphi_n = \varphi_{ns} - \frac{1}{\pi\nu\tan\varphi_s}\left(\frac{2\pi\tilde{l}}{\lambda}\kappa^2 + d_2\right)\varepsilon_n^3 + O\left(\varepsilon_n^4\right), \tag{8}$$

$$E_n = E_{ns} - \frac{\Delta E_s}{2\pi\nu}\left(\frac{2\pi\tilde{l}}{\lambda}\kappa^2 + d_2\right)\varepsilon_n^2 + O\left(\varepsilon_n^3\right), \tag{9}$$

where we denoted $\kappa = mc^2/\Delta E_s$ and m is the particle rest mass. The coefficient

$$d_2 = \frac{16\pi^2}{\lambda^3\nu^2}\left[\frac{1}{B_0^2}\int_0^{z_B}(A(z))^2\,dz - \frac{1}{3B_0^3}\left(A(z_B)\right)^3\right] \tag{10}$$

and the effective drift space length $\tilde{l} = l + \Delta z_0$, where $\Delta z_0 = 2z_B - 2A(z_B)/B_0$, characterize the concrete magnetic field profile. In their definition enters z_B, an arbitrary point in the region of constant field, i.e. $z_B > z_0$. It can be proved that in fact the values of d_2 and Δz_0, and therefore the final result, do not depend

1-4244-0917-9/07/$25.00

on the particular choice of z_B. One can easily show that $z_{\max}(E) = R(E) + \Delta z_0 / 2$. We would like to note that following a certain algorithm developed in [4] terms of higher orders in ε_n in expansions (8), (9) can be calculated.

Solution (8), (9) defines the *synchronous trajectory* corresponding to the asymptotic synchronous phase φ_s in the case of RTMs with low energy injection. This notion was first introduced in Ref. [1] and is defined as the particle with initial conditions (φ_0, E_0) such that in the limit $n \to \infty$ it approaches the asymptotic (ultra-relativistic) synchronous particle with the phase space coordinates (φ_s, E_{ns}), i.e. $\varphi_n (\mathrm{mod}\ 2\pi) \to \varphi_s$, $E_n \to E_{ns}$. The phase shift of the synchronous particle follows the well determined pattern described by Eq. (8). We would like to note that a general phase trajectory in addition to the terms in Eqs. (8), (9) includes synchrotron oscillations as it is described in Refs. [1,2].

FIXING THE INJECTION PHASE IN CASE OF END MAGNETS WITH THE INVERSE POLE

The formulas obtained here can be applied to the determination of the RTM parameters, in particular of the injection phase φ_0 of the synchronous trajectory. To fix such trajectory two parameters must be adjusted. As such parameters of tuning we will take φ_0 and the distance between the end magnets l. This is a common situation in RTM designs since the injection energy E_0 is usually fixed by the electron gun and accelerating structure voltages. For the sake of illustration let us suppose that formulas (8),(9) are exact enough already after the first orbit (generalization to higher orbits is straightforward). The phase and energy (φ_1, E_1) on one hand are given by Eqs. (8),(9) and on the other hand by recurrence relations (6) with $n = 0$. These relations define a system of two equations with respect to φ_0 and l. Here ε_1 and $E_{1,s}$ must be understood as functions of l. Details of this procedure are explained in Ref. [4]. As an illustration let us consider two examples of calculation of the synchronous trajectory for an RTM with λ=5 cm, ν=1, $\Delta E_{\max}$=2.08 MeV, φ_s=16° and with the end magnet with inverse pole (see Fig. 1) whose magnetic field is described by

$$B(z) = B_0\left[1 + \exp\left(-\frac{z - z_2}{h_0}\right)\right]^{-1} - B_1 \cosh^{-2}\left(\frac{z - z_1}{2h_0}\right) \qquad (11)$$

or

$$A(z) = B_0 h_0 \ln\left(1 + \exp\left(\frac{z - z_2}{h_0}\right)\right) - 2B_1 h_1\left[\tanh\left(\frac{z - z_1}{2h_1}\right) + 1\right] \qquad (12)$$

for $0 < z < z_0$. In accordance with our suppositions

$A(z) = 0$ for $z < 0$ and

$A(z) = B_0 z - (B_0 z_2 + 4B_1 h_1)$ for $z > z_0$

Let us note that magnetic systems with the main and inverse poles are used quite widely in modern RTMs. In what follows we will use the values $h_0 = h_1 = 0.1$ cm, $z_1 = 0.75$ cm, $z_2 = 2$ cm, $B_0 = 0.83\mathrm{T}, B_1 = 0.3B_0$ of the parameters of field (11) and potential (12).

Example 1. E_0=12.536 MeV, $\mu = 17$

The solution gives $(l/\lambda)_{th} = 4.12$ and $\varphi_{0,th} = 40.27^\circ$. In this case $\varepsilon_1 \approx 0.14$.

Example 2. E_0=2.536 MeV, $\mu = 12$.

The values of l/λ and φ_0 obtained from the system of equations are $(l/\lambda)_{th} = 3.99$, $\varphi_{0,th} = 17.90°$, so that the value $\varepsilon_1 \approx 0.4$.

For the sake of comparison we present the results from Ref. [1] for the same values of E_0 and μ but in the case of an end magnet with constant field $B = 0.83\mathrm{T}$ without inverse pole and fringe field.

Example 1: $(l/\lambda)_{th} = 4.86$, $\varphi_{0,th} = 15.60°$

Example 2: $(l/\lambda)_{th} = 4.84$, $\varphi_{0,th} = -0.79°$

CONCLUDING REMARKS

We have derived analytic formulas that describe the phase slip effect in the RTM longitudinal dynamics in the case of arbitrary magnetic field profile in the end magnets. We have shown that being applied to the design of RTMs they allow to define the generalized synchronous trajectory, at least as a first approximation. The accuracy of the formulas depends on the value of the parameter ε_n at the orbit where the analytical method is applied. Their precision can be increased by including terms with higher powers in ε_n. We would like to note that, in fact, the results are scale invariant, i.e. dimensionless combinations l/λ, z_1/λ, etc. only. Further details will be published elsewhere.

REFERENCES

[1] Yu.A. Kubyshin et al. "Analytic Study of Longitudinal Dynamics in Race-Track Microtrons", EPAC'06, Edinburgh, June 2006, p. 2008.
[2] Roy E. Rand, *Recirculating Electron Accelerators*, Harwood Academic Publishers, New York, 1984.
[3] V.G. Gevorkyan et al., *RTMTRACE*, preprint VINITI 678-88 (1988).
[4] Yu.A. Kubyshin, B.S. Ishkhanov, A.V. Poseryaev, V.I. Shvedunov, SINP Preprint 3/802 (2006).