A VARIATIONAL PRINCIPLE APPROACH TO THE EVOLUTION OF SHORT-PULSE LASER PLASMA DRIVERS*

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Abstract

The attractiveness of variational principle approaches for obtaining good approximations to complicated problems is well established. Motivated by this fact, we have developed a variational principle approach to the evolution of short-pulse laser-plasma accelerator drivers. We start with an action of the form $\partial_\tau a << k_0 a$ where the Euler-Lagrange equations of L, the Lagrangian density, give the well established coupled equations of short-pulse interactions, in the weakly relativistic regime:

$$(\nabla_\perp^2 - 2\frac{\partial^2}{\partial\psi\partial\tau} - 2ik_0\frac{\partial}{\partial\tau})\, a = (1-\phi)\, a$$

$$(\frac{\partial^2}{\partial\psi^2} + 1)\,\phi = \frac{|a|^2}{4}$$

We substitute appropriate trial functions for a and ϕ into S and carry out the $\int dx_\perp$ integration. The Euler-Lagrange equations of the reduced Lagrangian density provide coupled equations for the trial function parameters, i.e., spot sizes, amplitude, phase, radius of curvature and centroids for both a and ϕ. We present an analysis in the paraxial regime, where the $\partial_\psi\partial_\tau a$ term is neglected.

1 MOTIVATION

Understanding the evolution of short-pulse high-intensity lasers as they propagate through underdense plasmas is essential for the successful development of some plasma accelerator [1] and radiation schemes [2]. Research during the past few years has resulted in the identification of numerous instabilities such as envelope self-modulation [3], where the spot size of the laser becomes unstable, and hosing [4, 5], where the centroid of the laser becomes unstable. To study these instabilities, it is desirable to obtain differential equations for the evolution of the macroscopic quantities that characterize the beam profile, such as the spot size, amplitude, phase, radius of curvature and centroid. There are various methods of obtaining such equations, which include the variational method, as well as moment methods [6, 7], and the Source Dependent Expansion technique [8]. Here, we use a variational approach similar to that used by Anderson and Bonnedal in their study of self-focusing [9], and extend the variational technique to a coupled set of equations so as to include Raman related instabilities.

We start with the two coupled equations for the density perturbation and normalized vector potential, valid in the weakly relativistic regime, $|a|^2<<1$:

$$(\frac{\partial^2}{\partial t^2} - c^2\nabla^2)\,\vec{a} = 4\pi c\,\vec{J}_\perp = -\omega_p^2\frac{n}{n_0\gamma}\vec{a}$$

$$= -\omega_p^2(1 + \delta n - \frac{|a|^2}{2})\,\vec{a}$$

$$(\frac{\partial^2}{\partial t^2} + \omega_p^2)\,\delta n = c^2\nabla^2\frac{|a|^2}{2}$$

where $\delta n \equiv (n-n_0)/n_0$. Then, we normalizing all time dimensions to ω_p^{-1}, space dimensions to $k_p^{-1} \equiv c/\omega_p$, and make a coordinate transformation to the normalized light frame variables ($\psi \equiv t - z$, $\tau \equiv z$). Upon making the substitution $a \rightarrow a\exp[Ik_0\psi]$, switching to the potential $\phi \equiv \delta n - |a|^2/4$, making the envelope approximation $\partial_\tau a << k_0 a$, noting that $\vec{a}$ is unidirectional and describable by a single scalar, and dropping derivatives of $|a|^2$ as slow, we arrive at the well known set of equations:

$$(\nabla_\perp^2 - 2\frac{\partial^2}{\partial\psi\partial\tau} - 2ik_0\frac{\partial}{\partial\tau})\, a = (1-\phi)\, a$$

$$(\frac{\partial^2}{\partial\psi^2} + 1)\,\phi = \frac{|a|^2}{4}$$

2 THE VARIATIONAL APPROACH

Our approach to the analysis of these equations is to first obtain a Lagrangian density for the coupled equations. Then, we choose trial functions for a and ϕ which contain descriptive parameters that depend upon (ψ, τ). We substitute these trial functions into the action, and perform the integration across the transverse coordinates to obtain a reduced Lagrangian density. In this reduced form of the action integral, the parameters

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of the trial function represent a new set of dependent variables. In essence, by choosing an approximate form for the transverse beam profile, we replace the infinite degrees of freedom in the transverse directions with a finite number which represent the macroscopic characteristics of the distribution. Then, varying the action with respect to the new dependent variables yields a set of differential equations for the trial parameters.

We now carry out this procedure. We start by noting that the Lagrangian density for this set of equations is:

$$L = \vec{\nabla}_\perp a \cdot \vec{\nabla}_\perp a^* - ik_0(a\,\partial_\tau a^* - a^*\partial_\tau a) - (\partial_\psi a\,\partial_\tau a^* + \partial_\psi a^*\partial_\tau a) - 2(\partial_\psi \phi)^2 + 2\phi^2 - (\phi - 1)|a|^2$$

where the corresponding action is $S = \int dx_\perp d\psi\, d\tau\; L$. It can be readily verified that our starting equations are the result of varying the action with respect to a, a^*, and ϕ. We choose the following trial functions for a, and ϕ:

$$a = A(\psi,\tau)\; e^{i\,\mathrm{k_x}(\psi,\tau)\,\tilde{x}_a}\; e^{i\,\mathrm{k_y}(\psi,\tau)\,\tilde{y}_a} \times e^{-(1-i\alpha_x(\psi,\tau))\frac{\tilde{x}_a^2}{w_{xa}(\psi,\tau)^2}}\; e^{-(1-i\alpha_y(\psi,\tau))\frac{\tilde{y}_a^2}{w_{ya}(\psi,\tau)^2}}$$

$$\phi = \Phi(\psi,\tau)\, e^{-2\left(\frac{\tilde{x}_\phi^2}{w_{x\phi}(\psi,\tau)^2} + \frac{\tilde{y}_\phi^2}{w_{y\phi}(\psi,\tau)^2}\right)}$$

where $\tilde{x}_a \equiv x - x_a(\psi,\tau)$, $\tilde{y}_a \equiv y - y_a(\psi,\tau)$, $\tilde{x}_\phi \equiv x - x_\phi(\psi,\tau)$, $\tilde{y}_\phi \equiv y - y_\phi(\psi,\tau)$, and the amplitude A is a complex amplitude such that $A(\psi,\tau) = \sqrt{\xi(\psi,\tau)}\, e^{i\chi(\psi,\tau)}$. Inserting these trial functions into the action and performing the $\int dx_\perp$ integration yields the following reduced action principle:

$$\begin{aligned} S = \int d\psi\, d\tau \Big[& P\Big(\frac{\mathrm{k_x^2}}{2} + \frac{\mathrm{k_y^2}}{2} + \frac{(1+\alpha_x^2)}{2w_{xa}^2} + \frac{(1+\alpha_y^2)}{2w_{ya}^2} \\ & - k_0(\partial_\tau\chi - k_x\,\partial_\tau x_a - k_y\,\partial_\tau y_a)\Big) \\ & - P\,k_0\Big(\frac{w_{xa}^2}{4}\partial_\tau\Big(\frac{\alpha_x}{w_{xa}^2}\Big) + \frac{w_{ya}^2}{4}\partial_\tau\Big(\frac{\alpha_y}{w_{ya}^2}\Big)\Big) \\ & + \frac{1}{2}\,(P + w_{x\phi}w_{y\phi}\,\Phi) \\ & - \frac{w_{x\phi}w_{y\phi}\,\Phi\,P\; e^{-2\left(\frac{(x_a - x_\phi)^2}{(w_{xa}^2 + w_{x\phi}^2)} + \frac{(y_a - y_\phi)^2}{(w_{ya}^2 + w_{y\phi}^2)}\right)}}{\sqrt{(w_{xa}^2 + w_{x\phi}^2)(w_{ya}^2 + w_{y\phi}^2)}} + \ldots \end{aligned}$$

$$\begin{aligned} \ldots & - \Big(\frac{w_{x\phi}w_{y\phi}}{2}(\partial_\psi\Phi)^2 + \frac{\partial_\psi(\Phi^2)}{4}\partial_\psi(w_{x\phi}w_{y\phi}) \\ & + \frac{\Phi^2}{4}(\partial_\psi w_{x\phi})(\partial_\psi w_{y\phi}) + \frac{3}{8}\Phi^2\Big(\frac{w_{y\phi}}{w_{x\phi}}(\partial_\psi w_{x\phi})^2 \\ & + \frac{w_{x\phi}}{w_{y\phi}}(\partial_\psi w_{y\phi})^2\Big) + \Phi^2\Big(\frac{w_{y\phi}}{w_{x\phi}}(\partial_\psi x_\phi)^2 \\ & + \frac{w_{x\phi}}{w_{y\phi}}(\partial_\psi y_\phi)^2\Big) \end{aligned}$$

Varying the action with respect to χ, α_x, α_y, $\mathrm{k_x}$, and $\mathrm{k_y}$ yields the equations:

$$\begin{aligned} &\delta\chi: && \partial_\tau P = 0 \quad \text{(Power Conservation)}, \\ &\delta\alpha_x: && \alpha_x = -\frac{k_0}{4}\partial_\tau(w_{xa}^2), \\ &\delta\alpha_y: && \alpha_y = -\frac{k_0}{4}\partial_\tau(w_{ya}^2), \\ &\delta k_x: && k_x = -k_0\,\partial_\tau x_a, \\ &\delta k_y: && k_y = -k_0\,\partial_\tau y_a \end{aligned}$$

We can use these equations to eliminate χ, α_x, α_y, $\mathrm{k_x}$, and $\mathrm{k_y}$ from the Lagrangian since χ is an ignorable coordinate, and the other equations are generated by variations of the action with respect to the same variables being solved for. Eliminating these variables yields the following simplified form of the Lagrangian:

$$\begin{aligned} L = P\Big[& \frac{1}{2}\Big(\frac{1}{w_{xa}^2} + \frac{1}{w_{ya}^2}\Big) - \frac{k_0^2}{8}\big((\partial_\tau w_{xa})^2 + (\partial_\tau w_{ya})^2 \\ & + 4(\partial_\tau x_a)^2 + 4(\partial_\tau y_a)^2\big)\Big] + \frac{1}{2}\,(w_{x\phi}w_{y\phi}\,\Phi \\ & - \frac{w_{x\phi}w_{y\phi}\,\Phi\,P\; e^{-2\left(\frac{(x_a - x_\phi)^2}{(w_{xa}^2 + w_{x\phi}^2)} + \frac{(y_a - y_\phi)^2}{(w_{ya}^2 + w_{y\phi}^2)}\right)}}{\sqrt{(w_{xa}^2 + w_{x\phi}^2)(w_{ya}^2 + w_{y\phi}^2)}} \\ & - \Big(\frac{w_{x\phi}w_{y\phi}}{2}(\partial_\psi\Phi)^2 + \frac{\partial_\psi(\Phi^2)}{4}\partial_\psi(w_{x\phi}w_{y\phi}) \\ & + \frac{\Phi^2}{4}(\partial_\psi w_{x\phi})(\partial_\psi w_{y\phi}) + \frac{3}{8}\Phi^2\Big(\frac{w_{y\phi}}{w_{x\phi}}(\partial_\psi w_{x\phi})^2 \\ & + \frac{w_{x\phi}}{w_{y\phi}}(\partial_\psi w_{y\phi})^2\Big) \\ & - \frac{\Phi^2}{c^2}\Big(\frac{w_{y\phi}}{w_{x\phi}}(\partial_\psi x_\phi)^2 + \frac{w_{x\phi}}{w_{y\phi}}(\partial_\psi y_\phi)^2\Big)\Big) \end{aligned}$$

In this form, P is treated explicitly as a constant.

Variation of this Lagrangian with respect to the remaining parameters yields the desired set of differential equations for their evolution. These equations can then be used to study the stability of the beam profile.

3 SYMMETRIC ENVELOPE SELF MODULATION AND SELF-FOCUSING

To demonstrate the utility of this approach, we now look at the specific case of the symmetric envelope self modulation instability [3]. Here, we are only interested in the evolution of the spot sizes, with the x and y dimensions identical, i.e. $w_{xa} = w_{ya} \equiv w_a$, $w_{x\phi} = w_{y\phi} \equiv w_\phi$. Making these replacements, setting the centroids to 0, and then varying the resulting action with respect to the remaining parameters - Φ, w_a, w_ϕ - yields:

$$\delta w_a : \frac{k_0^2}{2}\partial_\tau^2 w_a - \frac{2}{w_a^3} + \frac{\Phi w_\phi^2 w_a}{(w_a^2 + w_\phi^2)^2} = 0$$

$$\delta w_\phi : (\partial_\psi \Phi)^2 - \frac{1}{2}\partial_\psi^2(\Phi^2) - 2\partial_\psi(\Phi^2 \partial_\psi w_\phi) + \Phi - \frac{P\Phi w_a^2}{(w_a^2 + w_\phi^2)^2} = 0$$

$$\delta \Phi : 2\partial_\psi(w_\phi^2 \partial_\psi \Phi) + \Phi\partial_\psi^2(w_\phi^2) - 4\Phi(\partial_\psi w_\phi)^2 + w_\phi^2 - \frac{P w_\phi^2}{w_a^2 + w_\phi^2} = 0$$

If we reduce these equations to the limit of no ψ dependence, we obtain the self-focusing equations:

$$w_\phi = w_a , \quad \Phi = \frac{P}{4w_a^2} = \frac{A^2}{4}$$

$$\partial_\tau^2 w_a - \frac{4}{k_o^2 w_a^3}\left(1 - \frac{P}{32}\right) = 0$$

From which we obtain the well known critical threshold for self-focusing, $P/P_{crit} = a_0^2 w_a^2/32$ [9, 10]. For $P = P_{crit}$, we have a matched, stationary beam profile, i.e., w_a is a constant.

We now examine the stability of this equilibrium solution. Linearizing the equations using a matched beam as the 0^{th} order solution yields:

$$\partial_\tau^2 w_{a1} + \frac{8}{k_0^2 w_0^4}\left(3 - \frac{P}{P_{crit}}\right) w_{a1} = -\frac{1}{k_0^2 w_0}\Phi_1$$

$$(\partial_\psi^2 + 1)\Phi_1 = -\frac{16}{w_0^3}\frac{P}{P_{crit}} w_{a1}$$

$$(\partial_\psi^2 + 1) w_{\phi 1} = w_{a1}$$

The first two of these equations show that Φ_1 and w_{a1} behave as coupled harmonic oscillators, which resonantly drive each other, and result in an envelope self-modulation instability.

A more general stability analysis can be done in which the centroids are allowed to evolve, and the spot sizes are allowed to be different in the x and y directions. This more general analysis results in 2 more instabilities: a hosing instability [4, 5, 11], wherein the centroids for a and ϕ resonantly drive each other, and an anti-symmetric envelope self modulation instability, wherein the linearized spot size variables in the x and y directions, for both a and ϕ, are equal and opposite (as opposed to the symmetric case previously analyzed, where the x and y spot sizes are identically equal). A comprehensive analysis of these instabilities has been performed, and will be described in future work.

4 SUMMARY

We have developed a variational principle approach for studying the evolution of short-pulse laser-plasma accelerator drivers. The approach is shown to reproduce previous results, e.g., relativistic self-focusing and spot size self-modulation. It is also useful for describing instabilities such as asymmetric self-focusing, asymmetric spot size self-modulation, as well as new long-wavelength regimes for hosing and spot-size self-modulation. We are currently extending the analysis to fully non-linear driver amplitudes.

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