CONSEQUENCES OF THE DIRECT SPACE CHARGE EFFECT FOR DYNAMIC APERTURE AND BEAM TAIL FORMATION IN THE LHC

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Abstract

The direct space charge detuning at the centre of a nominal LHC proton bunch is about 10^{-3} at injection energy. Owing to their slow synchrotron motion, particles with small betatron amplitudes sample a varying electric charge density within the bunch and thus experience a tune modulation at twice the 66 Hz synchrotron frequency. In conjunction with magnet nonlinearities, such tune modulation may give rise to particle diffusion to large betatron amplitudes and eventually to particle loss. Coherent quadrupole oscillations, caused, *e.g.*, by injection mismatch, resonantly perturb the single-particle motion via the space-charge force, and can cause a rapid growth in the transverse amplitude. Using the results of numerical and analytical investigations, we show that these space-charge effects alone will not affect the LHC performance.

1 INTRODUCTION

With an injection energy of 450 GeV, space charge effects at the LHC would appear to be negligible. However, there are two mechanisms by which the direct space charge force could cause a modulation of the betatron tunes, and, thereby, degrade the dynamic aperture or generate large-amplitude beam tails: First, synchrotron oscillation in conjunction with the longitudinal bunch profile induces a tune modulation at twice the synchrotron frequency, with a modulation amplitude comparable to the space-charge tune shift. Second, if the beam is injected with a betatron mismatch, the oscillation of the beam size, prior to filamentation, will also result in a tune modulation, at twice the betatron frequency.

2 SPACE CHARGE FORCE

Considering an optical lattice with vertical normalized quadrupole gradient $K_y(s)$, the equation of vertical motion for a single particle can be written as $y'' = -K_y y + F_{sc,y}(x, y)$, where a prime denotes the derivative with respect to the longitudinal position s, x is the horizontal coordinate, and $F_{sc,y}$ the vertical space charge force. In linear approximation (for $y \ll \sigma_y$, $x \ll \sigma_x$, where σ_y and σ_x are the transverse rms beam sizes), this force is

$$F_{sc,y}(x, y) \approx \frac{2F_{dist} r_p \lambda(z)}{\gamma^3 \sigma_y(\sigma_x + \sigma_y)}\, y \qquad (1)$$

The function $\lambda(z) = N_b \exp(-z^2/(2\sigma_z^2))/(\sqrt{2\pi}\sigma_z)$ denotes the longitudinal distribution assumed to be Gaussian, N_b the bunch population, and γ the Lorentz factor. The factor F_{dist} depends on the transverse distribution: $F_{dist} = 1$ for a Gaussian, 1/2 for a KV (uniform) distribution [1], and 2/3 for a parabolic distribution [2]. The space-charge tune shift at the bunch center ($z = 0$) is

Table 1: LHC injection parameters [3].

parameter	symbol	value
circumference	C	26.7 km
beam energy	E_b	450 GeV
particles per bunch	N_b	10^{11}
normalized transv. emittance	ϵ_N	3.75 μm
rms bunch length	σ_z	13 cm
average beta function	β	90 m
rms energy spread	δ_{rms}	4.7×10^{-4}
synchrotron tune	Q_s	0.006

$$\Delta Q_y \approx \frac{C r_p F_{dist} N_b}{4\pi\gamma^2 \epsilon_N \sqrt{2\pi}\sigma_z} \qquad (2)$$

where β designates the average vertical beta function, and ϵ_N the transverse normalized emittance. Using LHC parameters (Table 1), we find $\Delta Q_y \approx 1.1 \times 10^{-3}\ F_{dist}$.

At larger amplitudes, the space-charge force is nonlinear. For a flat Gaussian distribution this force can be calculated in terms of the complex error function [4]. Expressions for the parabolic and for the uniform distribution can be found in Ref. [5]. Via $\lambda(z)$ the space charge force, Eq. (1), depends on the coordinate z. Therefore, a particle performing large synchrotron oscillations and small transverse oscillations will experience a modulation of its betatron tune of amplitude ΔQ_y, due to the harmonic variation in z. A tune modulation of comparable magnitude will also occur in the horizontal plane.

3 SYNCHROTRON OSCILLATIONS

The effect of the tune modulation due to space-charge force and synchrotron oscillations, was studied by particle tracking with MAD [6], for the same LHC model as described in [8]. In the simulation, we launched twin particles at different transverse initial amplitudes. From their separation as a function of time we computed the Lyapunov exponent, which is a measure of potential instability. The longitudinal amplitude was chosen as $1.6\sigma_\delta$ (equal to $\delta = 7.5 \times 10^{-4}$, or three quarters of the rf bucket half size). The space charge force was modeled as a tune modulation of amplitude up to ΔQ_y at twice the synchrotron frequency, which was generated by harmonically varying the strength of the two main quadrupole families. In general, tune modulation is known to increase the chaotic (unstable) region of phase space [7]. However, comparing the Lyapunov exponents obtained with and without the additional tune modulation reveals no significant effect for these parameters (see Fig. 1). The observed weak effect is consistent with the results of chromaticity scans [8], and may be attributed to the absence of simultaneous modulation at other frequencies.

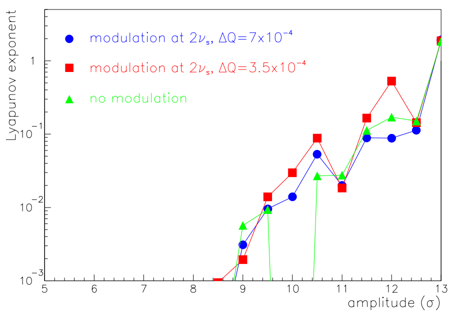

Figure 1: Lyapunov exponent computed by tracking over 5×10^4 turns as a function of the initial betatron amplitude, comparing cases with and without an additional tune modulation that represents the effect of space charge and synchrotron oscillations.

4 QUADRUPOLE OSCILLATIONS

The equation governing the evolution of the second moment $\sigma_y = \sqrt{< y^2 >}$ (the angular brackets denote an average over the beam distribution) follows from the single-particle equation of motion [9]:

$$\sigma_y'' + K_y \sigma_y - \frac{\epsilon_{y,rms}^2}{\sigma_y^3} = \frac{2 F_{dist} r_p \lambda(z)}{\gamma^3 (\sigma_x + \sigma_y)} \quad (3)$$

The term on the right hand side is due to the space charge, and we have only taken into account the linear component of the space-charge force. Consider a beam injected with a vertical (betatron) mismatch of amplitude $\Delta\sigma_y(0)$ at $s = 0$. The total beam size is the sum of the unperturbed (matched) value σ_{y0} and the perturbation: $\sigma_y = \sigma_{y0} + \Delta\sigma_y$. The equation of motion for the perturbed beam size is obtained by linear expansion around σ_{y0}:

$$\Delta\sigma_y'' + \left(K_y + 3 \frac{\epsilon_{y,rms}^2}{\sigma_{y0}^4} + \frac{2 F_{dist} r_p \lambda(z)}{\gamma^3 (\sigma_x + \sigma_{y0})^2} \right) \Delta\sigma_y = 0 \quad (4)$$

For simplicity, we now assume a smooth focusing, replacing K_y by $1/\beta^2$. Also using the relation $\sigma_{y0}^2/\beta = \epsilon_{y,rms}$, which holds for a matched beam, and neglecting the space-charge induced tune shift, we find the approximate solution $\Delta\sigma_y(s) \approx \Delta\sigma_y(0) \cos 2s/\beta$. Next, we can insert the analytical solution for the oscillation of the beam rms size into the equation of motion for a single particle above, replace σ_y by $\sigma_{y0} + \Delta\sigma_y(s)$, and introduce the new 'time' unit $u = s/\beta$. We assume that the beam is perfectly matched in the other (horizontal) plane, that the variation $\Delta\sigma_y$ is small compared with the matched beam size, and, for simplicity also that $\sigma_{y0} \approx \sigma_{x0}$ (which we call σ_0). Neglecting the shift in betatron tune, we finally obtain

$$\frac{d^2 y}{du^2} = (-1 - D \cos 2u)\, y \quad (5)$$

where

$$D \equiv \frac{3}{2} \frac{F_{dist} r_p \beta^2 \lambda(z)}{\gamma^3 \sigma_0^2} \frac{\Delta\sigma_y(0)}{\sigma_0} \quad (6)$$

For LHC parameters and a Gaussian distribution: $D \approx 5 \times 10^{-5}\ \Delta\sigma_y(0)/\sigma_{y0}$. Equation (5) is Mathieu's equation. With Q denoting the total betatron tune, the tune modulation amplitude corresponding to D is $\Delta Q \approx \frac{1}{2} D Q \approx 0.0015\, \Delta\sigma_y(0)/\sigma_{y0}$. The solution of the Mathieu equation is of the Floquet type: $F_\nu(u) = e^{iu\nu} P(u)$, where $P(u)$ is a periodic function of period π. For our parameters, ν has an imaginary component, and one solution is exponentially growing (the other shrinking). The exponent ν can be determined numerically (see [12]). With 20% accuracy we find that $e^{i\pi\nu} \approx 1 + D$ over a wide range of parameter values (*e.g.*, for D between 10^{-5} and 10^{-1}). The growth per turn is $(1 + D)^{2Q} \approx (1 + 2DQ)$, and the exponential growth time $\tau \approx C/(c[\ln(1 + 2DQ)]) \approx C/(2cDQ)$, with C the circumference, and c the speed of light. This relation is illustrated in Fig. 2, for the LHC parameters $2DQ \approx 0.006 \Delta\sigma_y(0)/\sigma_{y0}$ (using $Q \approx 63$). With an initial mismatch of 50%, the exponential growth time is of the order of 40 ms, while for 25% mismatch it is 60 ms.

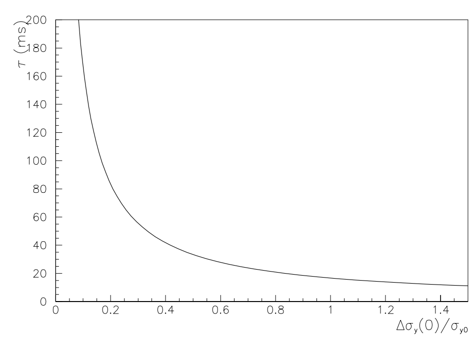

Figure 2: Exponential growth time as a function of the mismatch $\Delta\sigma_y(0)/\sigma_{y0}$. For the LHC 10 ms is about 100 turns.

Emittance growth and the generation of beam halo in proton linacs [10] and synchrotrons [11] has been studied by computing the trajectories of test particles inside and outside of a beam core whose dynamics is calculated independently. Following the same recipe, we numerically solved the single-particle equation of motion, using either the linear force of Eq. (1), or the full nonlinear force. The rms beam sizes σ_y, σ_x were modulated according to $\sigma_x \approx \sigma_y \approx \sigma_{y0} + \Delta\sigma_y(0)\ \cos 2s/\beta$. When the space charge kick for individual particles was calculated, we subtracted the linear force obtained for $\sigma_y = \sigma_{y0}$, since this would lead to the same tune shift for both the core and the individual test particles. Several hundred particles were uniformly distributed up to $\pm 2\sigma$ with random betatron phase. With the linear force of Eq. (1) the simulation yields a mean-amplitude growth rate of about $\tau \approx 35$ ms for $\Delta\sigma_y(0)/\sigma_{y0} = 50\%$, consistent with our analytical estimate. For the nonlinear force, the simulated growth rate is about 4 times slower (140 ms). The maximum amplitude over all test particles shows an oscillation whose phase is $\pi/2$ behind the core oscillation (each at twice the betatron frequency). Thus, the particle-core simulation is not self-consistent, since, in reality, the quadrupole oscillation would be affected by the growth of the individual particle amplitudes.

More realistic simulations can be performed with the program MAD, which was modified so that, at each lattice element, it applies a horizontal and vertical space-charge kick. The space-charge kick is calculated from the transverse rms spot size of a group of tracked macroparticles representing the beam. In Fig. 3 we depict typical results, that were obtained for a bunch population of $N_b = 10^{12}$, 10 times the nominal, and a linear space-charge force. The MAD simulation shows growing or shrinking betatron amplitudes depending on the phase of the betatron oscillation with respect to the mismatch. The growth stops, once the quadrupole oscillation has vanished.

Without systematic octupole field components, the quadrupole oscillation is damped mainly by filamentation due to the space-charge tune spread. Only considering the variation of the space-charge tune shift with longitudinal position, the time constant for this filamentation is

$$\tau_{sc} \approx \frac{1}{2\pi\sqrt{2}\sqrt{\frac{1}{\sqrt{3}} - \frac{1}{2}}\,\Delta Q_y} \approx \frac{1}{2.5\,\Delta Q_y} \qquad (7)$$

In our example, it evaluates to 36 turns, in good agreement with the simulation. Interestingly, since both damping and growth rates are proportional to the charge per bunch, the final emittance growth resulting from the space-charge force is independent of the bunch charge. For smaller bunch charges other filamentation mechanisms, *e.g.*, due to nonlinear magnetic fields, become effective, leading to an enhanced damping of the quadrupole oscillation (Fig. 4).

5 CONCLUSION

Tune modulation due to direct space-charge force along with synchrotron oscillations does not significantly increase the chaotic region in phase space. While a quadrupole-mode oscillation may cause fast amplitude growth rates, in practice this oscillation vanishes rapidly via filamentation due to space-charge induced tune spread and due to magnet nonlinearities. Thus, it is unlikely to result in serious emittance growth, for the LHC.

6 REFERENCES

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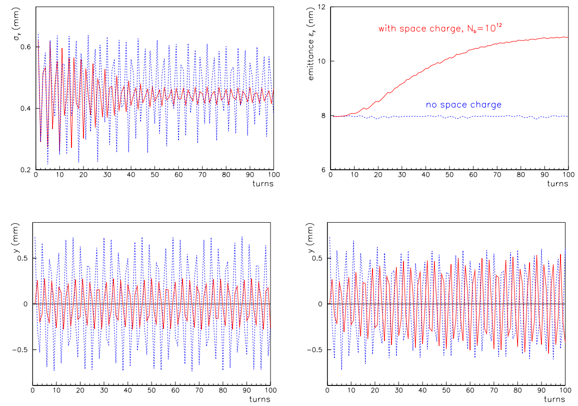

Figure 3: MAD tracking results using 1000 macroparticles, for an injection mismatch into the LHC of $\Delta\sigma/\sigma_{y0} \approx 0.5$ (or $\beta/\beta_0 = 2.6$). Top left: vertical beam size vs. time, with (solid) and without (dotted) space charge force; top right: emittance vs. time with (solid) and without (dotted) space charge; bottom left: vertical trajectories launched at 0.6σ of the injected beam with either initial offset (dotted) or initial slope (solid), without space charge; bottom right: same as the left figure but with space charge. The space-charge force was calculated in linear approximation, for $N_B = 10^{12}$. Magnet octupole components were not included.

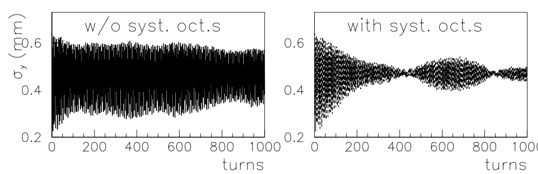

Figure 4: Vertical beam size vs. time, when the systematic octupole field components in the dipole magnets are corrected (left) or uncorrected (right).

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