

TO THE QUANTUM LIMITATIONS IN BEAM PHYSICS*

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Abstract

The conditions reviewed for degeneration in an electron bunch treated as a Fermi gas. Some comparison made for photoinjectors and storage rings. In some cases a quantum limitations are very close.

1 INTRODUCTION

In recent times some interest is growing to the beams with minimal emittance [1-3]. Schemes proposed which look like they even could reach the quantum limit in beam emittance. However, in some of these publications claimed that the minimal normalized emittance of the beam is defined by uncertainty principle only like $\gamma\epsilon_{\min} \equiv \lambda_c / 2$, where $\lambda_c = \hbar / mc \equiv 3.86 \cdot 10^{-11}$ cm – is a Compton wavelength [2,3]. Meanwhile it is well known that the *Fermi*-particles could not occupy the same states. This fact yields that for the beam with population equals N , the minimal emittance is N times bigger, than indicated above. This fact was illuminated first in [1].

2 CONDITION FOR DEGENERATION

Electron gas has a tendency to generate with lowering its temperature. If gas is fully degenerated, each state occupied by a pair of electrons having different spin orientation. Basically we have definition of degenerated state as one having temperature small compared with Fermi energy E_F [4]. The last is the highest one while states are tightly packed. We will consider the minimal emittance definition from two slightly different approaches. First uses the fact, that there are no free states in a system below E_F . This yields that the number of states equals to the number of the particles in the beam. This definition basically corresponds to assumption that the beam has zero temperature. We will simply calculate the number of allowed states. The second approach, indeed, uses the definition of the beam temperature in a moving system of reference. We compare the emittance restrictions obtained with these two approaches. Ultrarelativistic electron gas becomes more ideal in a Lab frame with increasing its energy due to cancellation of forces between the particles as $\approx 1/\gamma^2$. In principle, some amount of particles with opposite charge can be added to the beam moving along a straight line in a focusing channel [7].

Now let us estimate the minimal emittance from the compact packing approach. For the electrons in a volume V , the number of states with absolute magnitude of momentum in the interval from $p_{x,y,s}$ to $p_{x,y,s} + dp_{x,y,s}$ is

$$dn \cong 2 \frac{dp_x dp_y dp_s \cdot V}{(2\pi\hbar)^3}, \quad (1)$$

where factor 2 reflects two possibilities for spin statement. Let us suggest that all lower states are occupied up to the highest one. Then the number of the states equals to the number of the particles in the volume. The Fermi-momenta p_F defined as

$$N = \int dn \cong 2 \cdot \frac{4\pi}{3} \frac{p_F^3 \cdot V}{(2\pi\hbar)^3} \cong \frac{8\pi}{3} \frac{V}{\lambda_B^3}, \quad (2)$$

where $\lambda_B \equiv 2\pi\hbar / p_F$ is the length of the de-Broigle wave corresponding to the particle with Fermi energy. So the density N/V in degenerated Fermi-gas corresponds to the reverse cube of de-Broigle wavelength, corresponding to the Fermi momenta. In formula (1) for the number of states the variables (momenta and coordinate) are canonically conjugated. The particles in a focusing system behave more likely as a harmonic oscillator. Description of transverse motion with envelope function satisfy the requirements to be a canonically conjugated as

$$p_x p_y S_{\perp} \cong m c \gamma \sqrt{\epsilon_x / \beta_x} \cdot m c \gamma \sqrt{\epsilon_y / \beta_y} \cdot \sqrt{\epsilon_x \beta_x} \cdot \sqrt{\epsilon_y \beta_y}, \quad (3)$$

where $\beta_{x,y}$ – are the envelope function. The total number of electrons in these states can be estimated for uniform distribution from (1) using (3)

$$N = \int dn \cong \frac{2 p_x p_y \Delta p_{\parallel} S_{\perp} l_b}{(2\pi\hbar)^3} \cong \frac{2 \gamma \epsilon_x \gamma \epsilon_y \gamma l_b (\Delta p_{\parallel} / p_0)}{(2\pi\lambda_c)^3}, \quad (4)$$

$$\text{or } (\gamma \epsilon_x)(\gamma \epsilon_y)(\gamma \epsilon_s) \geq (\frac{1}{2})(2\pi\lambda_c)^3 N, \quad (5)$$

where $\gamma \epsilon_s = \gamma l_b (\Delta p_{\parallel} / p_0)$ – is an invariant longitudinal emittance, l_b – is the bunch length, $(\Delta p_{\parallel} / p_0)$ – the relative momentum spread in the beam, $\gamma \epsilon_x$ and $\gamma \epsilon_y$ – are the transverse horizontal and vertical emittances. We can also say, that the beam with the number of the particles N cannot have emittances lower than defined by (5) as all lower states are occupied. So in that sense formula (5) corresponds to *zero* beam temperature and

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could be treated as *absolute lower limit* for the beam emittances. The right side in (5) is tremendously small as $(2\pi\lambda_c)^3 \cong 1.4 \cdot 10^{-30} \text{ cm}^3$. So even for $N \cong 10^{10}$ the right side in (5) will be $\propto 10^{-20} \text{ cm}^3$.

Now let us estimate the minimal emittance from the requirement that the beam temperature is much less, than the Fermi energy. This condition is weaker, that (5). From (2) one can obtain

$$p_F = (3/8\pi)^{1/3} 2\pi\hbar\rho^{1/3}, \quad (6)$$

where $\rho \cong N/V$ – is a density in the rest frame. From (6) one can obtain, that the Fermi energy is

$$E_F = \frac{p_F^2}{2m} \cong (3\pi^2)^{2/3} \frac{\hbar^2}{2m} \rho^{2/3} \quad (7)$$

or $E_F = cp_F = (3\pi^2)^{1/3} \hbar c \rho^{1/3}$, (8)

if the particles in the rest frame are relativistic. The formulas (7), (8) represent the height of the well.

Using (7) or (8) one can find the condition for degeneration as

$$k_B T \leq \frac{(3\pi^2)^{2/3}}{2} mc^2 (\lambda_c^3 \rho)^{2/3},$$

or $k_B T \leq (3\pi^2)^{1/3} mc^2 (\lambda_c^3 \rho)^{1/3}$,

where $k_B \cong 1.38 \cdot 10^{-23} \text{ J}^\circ\text{K}$ – is Boltzmann's constant. The electron gas temperature T in a moving frame can be calculated using (3). The transverse momenta is invariant and the transverse kinetic energy is

$$\frac{p_x^2}{2m} + \frac{p_y^2}{2m} \cong \frac{1}{2m} m^2 c^2 \gamma^2 \frac{\varepsilon_x}{\beta_x} + \frac{1}{2m} m^2 c^2 \gamma^2 \frac{\varepsilon_y}{\beta_y}.$$

Full energy is a sum of kinetic and potential energy of motion in a focusing system. According to the virial theorem for harmonic oscillator the average potential energy equals to the kinetic one. So the temperature of the beam could be represented as the following

$$\frac{3}{2} N k_B T \cong N \cdot mc^2 \gamma [\gamma \varepsilon_x / \beta_x + \gamma \varepsilon_y / \beta_y + \beta^2 (\Delta p_{\parallel} / p_0)^2 / \gamma], \quad (9)$$

where $\beta^2 \cong 1 -$ is a square of normalized speed in the lab frame. Let us consider one example. In damping rings developed as injectors for future linear colliders, typical values are $\beta_{x,y} \cong 10\text{m}$, $l_b \cong 1\text{cm}$, $\Delta p/p \cong 5 \cdot 10^{-4}$, $\gamma \varepsilon_x \cong 3\text{cm}$, $\gamma \varepsilon_x \cong 3 \cdot 10^{-4} \text{ cm rad}$, $\gamma \varepsilon_y \cong 3 \cdot 10^{-6} \text{ cm rad}$.

This gives $\frac{3}{2} k_B T \cong mc^2 \gamma [3 \cdot 10^{-7} + 3 \cdot 10^{-9} + 4 \cdot 10^{-11}]$. One can see that despite the longitudinal emittance is the biggest one, the longitudinal temperature is the lowest one. This yields the possibility for redistribution the temperatures. Compare formula (9) with (8) one can obtain for relativistic electron gas

$$mc^2 \gamma \left[\frac{\gamma \varepsilon_x}{\beta_x} + \frac{\gamma \varepsilon_y}{\beta_y} + \frac{1}{\gamma} \left(\frac{\Delta p}{p_0} \right)^2 \right] \ll (3\pi^2)^{1/3} mc^2 (\lambda_c \rho)^{1/3},$$

where $\rho = \rho_0 / \gamma$, ρ_0 – is the density in the Laboratory frame. Neglecting the longitudinal temperature and supposing that, $\gamma \varepsilon_x \approx \gamma \varepsilon_y$ one can obtain the condition

$$(\gamma \varepsilon_x)^4 \leq \beta_x^2 \lambda_c^3 N / (\gamma^3 l_b). \quad (10)$$

Substitute here for estimation $\beta_x \approx l_b \approx 1\text{cm}$, $N \approx 10^{10}$, $\gamma \cong 6 \cdot 10^3$, one can obtain $\gamma \varepsilon_x \leq 2 \cdot 10^{-8} \text{ cm rad}$. This is estimation for the *maximal* possible transverse emittance required for degeneration. One can compare this figure with the one suggested for the Linear Collider. From the other hand this is not drastically lower, than for specially designed coolers, see lower.

3 REDUCTION OF DIMENSIONS

We suggested that the phase space corresponds to 3D motion in real space. So the particle could reach every point in phase space. Defining the speed of motion in the rest frame as $v \cong \sqrt{3k_B T / m}$, one can find the frequency of collision $f_{x,y}$ with the potential wall as

$$f_{x,y} \cong \frac{v_{x,y}}{a_{x,y}} \cong c \sqrt{\gamma (\gamma \varepsilon_{x,y}) / \beta_{x,y}} / \sqrt{(\gamma \varepsilon_{x,y}) \beta_{x,y} / \gamma} = \frac{c \gamma}{\beta_{x,y}} \quad (11)$$

like it must be if one transforms the frequency into moving frame. This frequency does not depend on emittance. For longitudinal motion one can estimate $f_s \cong v_{\parallel} / \lambda_b \cong c (\Delta p / p) / \lambda_b$. So one can see that the frequency of longitudinal oscillations is much smaller, than the transverse. In motion along the focusing channel the longitudinal motion is practically absent at all. So for a two-dimensional phase space

$$(\gamma \varepsilon_x)(\gamma \varepsilon_y) \geq (\frac{1}{2})(2\pi\lambda_c)^2 N. \quad (12)$$

Considerations [5] show that thermalization due to intrabeam scattering (also through the third agent, such as a resistive wall or parasitic cavity) can happen only when momentum compaction factor is *negative*.

From quantum mechanical consideration of the problem, for N fermions the wave function is fully anti-symmetric one. It could be written as a Slater's determinant. As the particles experience the motion in a kind of harmonic oscillator potential, the energy of each particle is proportional

$$E \cong E_{kin} + \frac{1}{2} \sum K_i x_i^2, \quad i = 1, 2, 3, \quad (14)$$

where K_i – is the effective rigidity for selected degree of freedom. Using methods of quantum mechanics one can obtain that the energy of ground state is

$$E_0 = \sum_{i=1,3} \hbar \omega_i \cdot \left(\frac{N}{2} + \sum_{n=1}^{N-1} n \right), \quad (15)$$

where n counts the states occupied and $\omega_i = \sqrt{K_i / m}$ – is a partial frequency. Expanding (15) one can obtain

$$E_0 = N^2 \cdot \sum \hbar \omega_i = \vec{J} \cdot \vec{\omega}, \quad i = 1, 2, 3, \quad (16)$$

where $|\vec{J}| = \hbar \cdot N^2$ could be treated as total action. So for one degree of freedom using (3) one can obtain

$$p_x \cdot \Delta x \cong mcN \sqrt{\epsilon_x / \beta_x} \cdot \sqrt{\epsilon_x \beta_x} \cong mc(\gamma \epsilon_x)N. \quad (17)$$

This could be called the total action for selected degree of freedom. Equating (16) to the value of action from (17) one can find the minimal emittance as

$$(\gamma \epsilon_x) \cong \hbar N^2 / mc \cdot N = \lambda_c N. \quad (18)$$

For a typical bunch population $N \approx 10^{10}$ (18) will be $(\gamma \epsilon_x) \cong 3.8 \cdot 10^{11} \cdot 10^0 = 0.38$ cm only. So we are coming to fundamental conclusion that one-dimensional system is always degenerated under real conditions. That means that there is no dynamic aperture for the particles in 1D. In that sense the claim that so called Mobius ring [6] brings particle motion into 1D and, hence, has some advantages, needs to be treated with cautions.

4 COOLERS AND BEAMS

One type of a cooler able to reach extreme emittances considered in [7]. Basically it contains the dipole wigglers and accelerating cavities installed in series so the average energy of the beam kept constant. It gives the emittances obtained after considerations the radiation dynamics for a single electron as

$$(\gamma \epsilon_x) \cong (\frac{1}{2}) \cdot \lambda_c \bar{\beta}_x (1 + K_x^2 / 2) K_x / \lambda \quad (19)$$

$$(\gamma \epsilon_y) \cong (\frac{1}{2}) \cdot \lambda_c \bar{\beta}_y K_x / \lambda, \quad (20)$$

where $\bar{\beta}_{x,y}$ —are averaged envelope functions in the wiggler. $K = eH_{\perp} \lambda / 2\pi mc^2$, H_{\perp} —is the magnetic field in the wiggler, λ —is the wiggler period. The last formulas together with the cooling time

$$\tau_{cool} \cong (\frac{3}{2}) \cdot (\lambda^2 / cK^2 \gamma), \quad (21)$$

defines the cooling dynamics under SR. One can see that equilibrium invariant emittances *do not depend on energy*. In addition, quantum equilibrium *vertical emittance* and the cooling time do not depend on the wiggler period. For successful operation of Laser Linear Collider the only $N \approx 10^6$ particles required [6]. If the $\bar{\beta}_y K_x / \lambda \approx 1$, then, formally, the emittance could be close to $(\gamma \epsilon_y) \approx (\frac{1}{2}) \lambda_c$. The last means that quantum limitations for the lowest emittance are important here. So one can treat this as emittance occupied by a phase trajectory of a *single* electron. For radiation in dipole wiggler the electron in a ground state remains radiating the photons. So to damp the transverse emittance any electron must re-radiate its full energy. This brings some final equilibrium emittance like (19), (20). Until this emittance is big compared with (4) there is no

confusion.

In [3] there was considered the radiation effects in a focusing channel. The last might be a sequence of focusing and defocusing lenses (what is basically a quadrupole wiggler). Here the electron in a ground state is not radiating. So there is no formal requirements for re-radiation of particle's full energy. However in this publication was claimed that the energy of the ground state is $E_0 = \hbar \omega (\frac{1}{2} + n)$, $n=0$ and the minimal emittance for the beam here could be as low as $(\gamma \epsilon_y) \approx (\frac{1}{2}) \lambda_c$. One can see that the factor associated with the number of the particles is missed here. One can see also that there are no advantages between dipole and quadrupole wiggler from the point of minimal emittance. We can add, that the quantum limitation occurs much earlier. In contrast, the cooling time for a traditional dipole wiggler is much smaller, than for quadrupole one.

In [8] there was considered the radiation of a relativistic electron in a solenoid. It was shown here that radiation here carries out the transverse energy only. In that sense the particle in a ground state is not radiated also. This means that electron can decrease its' emittance without re-radiation of its' full energy.

There was made a comparison of the lowest emittances in the RF photo-injectors and (5) in [9]. In some cases the quantum limitation (5) is close.

5 REFERENCES

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