THE PROBLEM OF DISPERSION MATCHING IN SPACE CHARGE DOMINATED BEAMS*

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Abstract

We recently proposed a new analytical model that incorporates the dispersion function into the framework of the rms envelope equations. Here we show how it can be used to achieve proper matching of space charge dominated beams. Comparison is done against a PIC code (WARP) simulations.

1 INTRODUCTION

Understanding the combined effect of space charge and an energy spread in a circular lattice and more specifically, inclusion of space charge forces in the calculation of the dispersion function has been the subject of a number of papers over the last few years $[1-6]$ The surge of interest has been motivated by the need of high intensity and quality beams in applications like the Spallation Neutron Source, the drivers for Heavy Ion Fusion or in studies of beam physics as in the Maryland Electron Ring [7]. In all these cases meeting optimal matching conditions (including the dispersion function) will be critical for a proper machine operation. In the presence of very high beam currents the calculation of the dispersion function is complicated in two regards: On the one hand space charge forces have to be taken into account to compute the dispersion function properly; on the other dispersion itself has a role in determining the space charge forces by affecting the beam shape. In [4] we developed an analytical model that describes both aspects of the interplay between space charge and dispersion for the case of continuous beams. The model is an extension of the standard rms evelope equations that are routinely used to solve the matching problem in straight channels. It consists of a set of three coupled differential equations for the horizontal and vertical envelopes and dispersion function. A distinctive feature is the dependence on a generalized rms emittance, which unlike the usual rms emittance is a linear invariant in the presence of an energy spread and bending magnets. A preliminary positive test on the validity of our generalized rms envelope equations was already discussed in [4]. The test was carried out against earlier calculations [5] involving a fully self-consistent analysis of beam distributions in smooth dispersive channels. Later [6] we compared the solutions of the new equations with simulations performed with a PIC code (WARP, [8]) for a periodic circular lattice consisting of FODO cells and bends. At that stage no effort was done to match the dispersion function at injection, while the envelopes were

matched using the standard rms envelope equations. Comparison showed good agreement over the scale of the first (depressed) betatron oscillation period. However, after the first betatron oscillation period one could observe relaxation phenomena driven by the space charge nonlinearities in the form of a damping in the oscillations of the horizontal rms emittance and an increase in the vertical rms emittance (see also [2]). These features are not captured by the generalized rms envelope equations (if used in conjunction with the assumption that the vertical rms emittance and the generalized rms emittance are invariant of the motion). Nevertheless we speculated that for the purpose of determining the matching conditions those equations should be adequate – as long as the matching is done over a length shorter than the betatron oscillation period. In this paper we finally test that guess and apply the generalized rms envelope equations to the problem of determining the matching conditions at both injection and extraction for the Maryland E-Ring. The results are reported in Section 3. First, in Section 2 we briefly review the form of the generalized rms envelope equations.

2 THE MODEL

The theory described in this paper applies to the dynamics of a continuous beam of charged particles confined in a dispersive channel and described by the Hamiltonian

$$H = \frac{1}{2}(p_x^2+p_y^2)+\frac{1}{2}(\kappa_x x^2+\kappa_y y^2)+\frac{\psi}{mv_z^2\gamma^3}-\frac{\delta}{\rho}+\frac{m^2c^4}{E_0^2}\delta^2,$$

where v_z is the longitudinal velocity, γ the relativistic factor, m and q the mass and charge of the beam particles. The self-force is described by the potential ψ. Each particle has a momentum $p = p_0(1 + \delta)$, with a relative derivation δ from the design momentum p_0; E_0 is the corresponding energy. The external focussing forces, described by the κ_x and κ_y, are assumed to be linear. In this model we also neglect chromatic effects (i.e. terms like $x^2\delta$ in the Hamiltonian). Finally, ρ is the local radius of curvature. Also, notice that the model (1) does not entail beam acceleration. The derivation of the generalized rms envelope equations from the Hamiltonian (1) can be carried out following steps similar to those needed to derive the usual rms envelope equations. The major differences are that i) there are two additional equations involving the moments $\langle x\delta\rangle$ and $\langle p_x\delta\rangle$; ii) the usual rms emittance is more conveniently replaced by a quantity [see below Eq. (3)] that is a linear invariant in the presence of an energy spread. Under the assumption that the beam density maintains an elliptical symmetry it can be shown [4] that the beam envelopes

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$\sigma_x = \sqrt{\langle x^2 \rangle}$, $\sigma_y = \sqrt{\langle y^2 \rangle}$ and the dispersion function D obey the system of equations:

$$\sigma_x'' = \frac{\epsilon_{dx}^2 + (\sigma_x \sigma_x' - DD'\langle \delta^2 \rangle)^2}{\sigma_x(\sigma_x^2 - D^2\langle \delta^2 \rangle)} - \frac{(\sigma_x')^2}{\sigma_x} + \kappa_x \sigma_x$$
$$+ \frac{K}{2(\sigma_x + \sigma_y)} + \frac{\langle \delta^2 \rangle}{\sigma_x}\left(\frac{D}{\rho} + D'^2\right),$$
$$\sigma_y'' = \frac{\epsilon_y^2}{\sigma_y^3} - \kappa_y \sigma_y + \frac{K}{2(\sigma_x + \sigma_y)}, \qquad (2)$$
$$D'' = -\left[\kappa_x - \frac{K}{2\sigma_x(\sigma_x + \sigma_y)}\right]D + \frac{1}{\rho},$$

where the generalized perveance is $K = 2I/(I_0\beta^2\gamma^3)$ with $I_0 = 4\pi\epsilon_0 mc^3/q$ (in MKS units), and $\sqrt{\langle \delta^2 \rangle}$ is the relative rms momentum spread. In this model the dispersion function D and its derivative are identified by $D(z) = \langle x\delta \rangle / \langle \delta^2 \rangle$ and $D'(z) = \langle p_x\delta \rangle / \langle \delta^2 \rangle$. Physically this identification is legittimate if before injection $\langle x\delta \rangle = \langle p_x\delta \rangle = 0$ (i.e. the pairs (x, δ) and (p_x, δ) are uncorrelated). We found this to be a natural way to extend the concept of dispersion function to include beams of self-interacting particles. The generalized rms emittance ϵ_{dx} appearing in the first equation in (2) is defined by

$$\epsilon_{dx}^2 = \epsilon_x^2 - \langle \delta^2 \rangle \langle [p_x D(z) - xD'(z)]^2 \rangle, \qquad (3)$$

where ϵ_x is the usual rms emittance. For matching purposes Eqs. (2) are of practical use if ϵ_{dx} does not change significantly as a result of the nonlinearities associated with space charge. One of our goals in checking Eqs. (2) against a PIC code is to establish the extent to which such an assumption is verified in practice.
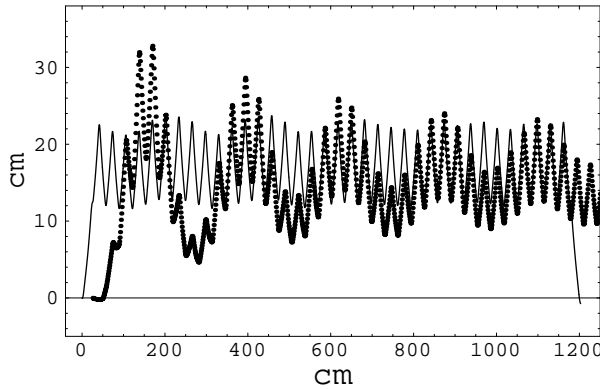
3   MATCHING DISPERSION

Figure 1: Dispersion function (in cm) with (solid line) and without (dots) dispersion matching; I=50 mA.

In this section we report the simulations done with the PIC code WARP to test various matching schemes worked out using the generalized rms envelope equations (2). The calculations presented here refer to a hard edge model of the

Maryland E-Ring [7]. The E-Ring is designed for circulation of a 10 KeV electron beam with a current in the proximity of 100 mA. It consists of 36 identical FODO cells, each one including a $10°$ dipole bend, for a circumference length of 11.52 m. In the calculations reported here injection into the Ring is accomplished by a dispersion matching module consisting of two dipoles and one quadrupole. Matching of the envelopes σ_x and σ_y can be carried out separately in the transport line between the electron source and the dispersion matching module using the standard rms envelope equations.
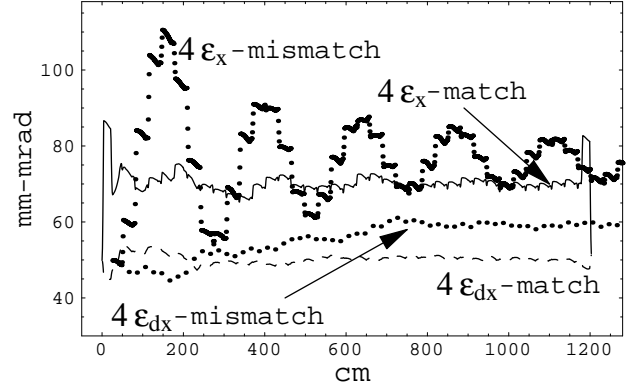

Figure 2: Horizontal effective emittances (in units of mm-mrad) and invariant (2) for the matched and mismatched cases; I= 50 mA.

The first set of pictures shows results from simulations obtained with the PIC code for a beam of $I = 50$ mA, corresponding to a tune depression of $\nu/\nu_0 = 0.29$ [the initial rms emittances are $\epsilon_x = \epsilon_y = 12.5$ mm-mrad and the energy spread is 300 eV (i.e. $\sqrt{\langle \delta^2 \rangle} = 0.015$); such an unrealistically large value has been chosen to emphasize the effect of dispersion in this study]. The beam evolution is followed from the beginning of the dispersion matching module through one turn of the E-Ring and through a dispersion matching module at extraction. The purpose of the extraction module is to bring the dispersion function and its derivative back to zero. Specifically, Fig. 1 shows the profile of the dispersion function for the matched case as calculated by WARP. For comparison we also report the dispersion function that one would get if no dispersion matching is done at injection (dots). In this case the dispersion function undergoes large oscillations that are rapidly damped by the space charge nonlinearities. In Fig. 2 the horizontal effective emittance is plotted for the matched and mismatched cases together with the rms invariant ϵ_{dx} defined in Eq. (3). One can observe that the damping in the oscillations displayed by the disperson function in the mismatched case (see Fig. 1) is reproduced here. As a result, the final value for the emittance that one can extrapolate for the mismatched case is larger (although not dramatically) than the corresponding value of the emittance in the case with dispersion matching (before extraction is done). Notice however that after extraction ϵ_x is brought back to a

value very close to the one it had initially. In other words, if maching is done the ϵ_x emittance growth appears to be almost completely reversible (at least over one turn). This is consistent with ϵ_{dx} remaining basically constant. On the other hand ϵ_{dx} increases noticeably in the mismatched case.
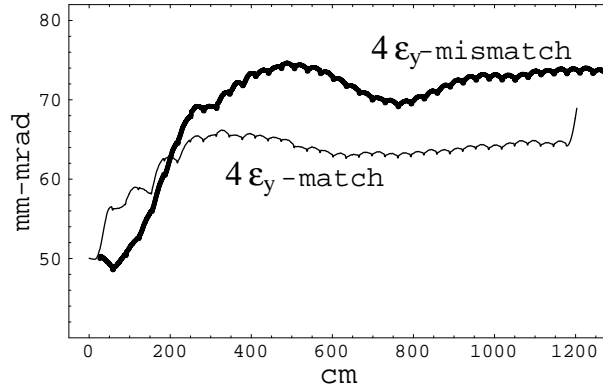

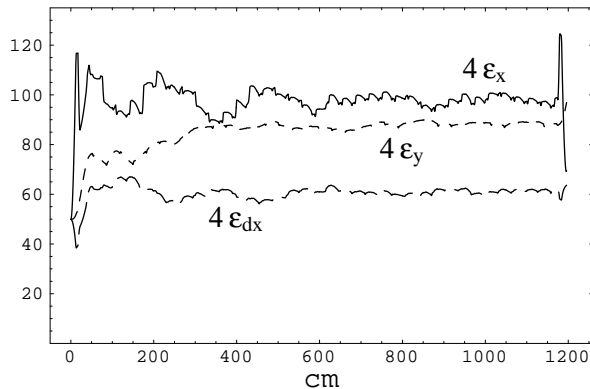Figure 3: Vertical effective emittances (in units of mm-mrad) for the matched and mismatched case; I=50 mA.

Figure 4: Horizontal and vertical effective emittances and invariant of Eq. (2) (in units of mm-mrad) for the matched and mismatched case; I= 100 mA.

Next, in Fig. 3 the evolution of the vertical emittances are reported. The rms ϵ_y emittance increases because of the nonlinear coupling with the horizontal motion induced by space charge. This effect is not captured by Eqs. (2). However, notice how a matching based on Eq. (2) nevertheless succeeds in reducing the amount of the y-emittance growth. The sharp growth that we can observe at extraction is due in part to the fact that at extraction the matching was done under the assumption that ϵ_y was the same as at injection. In Fig. 4 we show the evolution of the emittances for a case with larger current $I = 100$ mA (corresponding to a detuning $\nu/\nu_0 = 0.15$). In this case the matching is less efficient although still significant. In the last picture (Fig. 5) we report the case in which the matching is done using the equations proposed by A. Garren [1]. One can see that under the regime we are considering that model would lead to

an even more pronounced mismatch (A. Garren's equations coincide with Eqs. (2) in the zero energy spread limit).
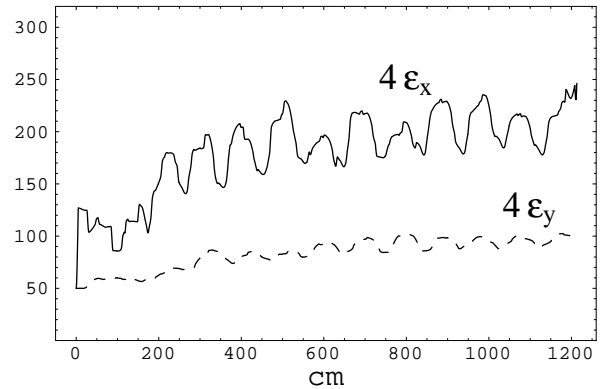

Figure 5: Horizontal and vertical effective emittances (in units of mm-mrad) for a matching done using a less accurate model; I= 50 mA.

4 CONCLUSIONS

The results reported in this paper show that use of the generalized rms envelope equations appears to be effective in achieving acceptable matching conditions for space charge dominated beams in the presence of an energy spread. Moreover, if the tune depression is not extreme the rms emittance growth in the horizontal plane due to dispersion seems to be to a large extent reversible. A measure of the non reversibility is given by the growth of the generalized emittance defined in Eq. (3).

5 AKNOWLEDGEMENTS

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