THE COUPLING IMPEDANCE OF A TOROIDAL BEAM TUBE WITH CIRCULAR CROSS SECTION*

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Abstract

In this paper, the longitudinal coupling impedance of a toroidal beam tube with circular cross section is derived in the frequency domain using the toroidal coordinate system. Exact, although coupled, differential equations for the azimuthal field components are obtained. An approximate solution, valid in the limit of small curvature, is then derived. Assuming extreme relativistic energies and a beam tube with perfectly conducting walls, one finds a closed-form expression for the purely reactive coupling impedance which at low mode numbers is dependent on the tube geometry but which at very large mode numbers approaches the free space impedance.

1 INTRODUCTION

In contrast to the incoherent synchrotron radiation, the coherent radiation has been so far of limited practical importance but could become important for machines with very short bunches. In any case, the study of the interaction of a circular beam with its environment, here described by the coupling impedance, is a topic of considerable theoretical interest and has been addressed by many accelerator physicists, most recently in the paper by Murphy et al[1] or in the book by Zotter and Kheifets,[2] where references to many relevant studies can be found. The complete expression for the coupling impedance due to coherent synchrotron radiation in *free space* is found in ref. 1; from this follow the Bonch-Osmolovski/Faltens/Laslett asymptotic approximation valid at mode numbers $n << n_c$ as well as the asymptotic approximation valid when $n >> n_c$, the latter given by

$$Z_n \approx Z_0 \gamma \left[\frac{\sqrt{\pi}}{4} \left(\frac{3}{2} \frac{n}{n_c} \right)^{1/2} e^{-n/n_c} + j\, \frac{4}{9} \frac{n_c}{n} \right]$$

with the critical mode number $n_c = (3/2)\,\gamma^3$ and the free space impedance $Z_0 = \mu_0\, c$ (= 1 in natural units used in this paper). Note that at very large n, the real part decreases exponentially and the free space coupling impedance becomes predominately inductive.

In accelerators/colliders the beam is enclosed in a beam tube of dimensions small compared to the bending radius. The coherent radiation by beam bunches of finite length has a spectrum mainly with wavelengths longer than the dimensions of the bunch and thus comparable to the transverse dimensions of the beam tube. Hence the presence of the walls, in this paper assumed perfectly conducting, changes the field configuration and leads to radiation shielding. The shielding effect has been analyzed for the simple geometries of a beam centered between parallel plates[1], [2] and in a toroidal tube with rectangular[3] or circular cross section.[4] The results show that the shielding effect is most pronounced at low mode numbers for which asymptotic approximations exist in closed form. The situation at large mode numbers is less clear, in particular with respect to the reactive part, since no closed form approximation seems to be known. Attempting to fill this gap, in this paper the general solution for a beam in a toroidal beam tube with circular cross section is derived using a perturbation method valid in the case of small curbature. From this solution, valid at all mode numbers, an approximate expression in closed form for the case of $n >> \gamma R/b$ is derived, which shows that the free space impedance is approached. Although the analysis is based on a specific beam tube the solution, the result is believed to be representative for any fully shielded geometry.

2 MAXWELL'S EQUATIONS IN TOROIDAL COORDINATES

The study of a toroidal beam tube with circular cross section suggests the use of the toroidal coordinate system (u, p, θ) even though the vector wave equation is not separable.[5] Toroidal coordinates are defined in terms of circular cylinder coordinates (ρ, θ, z) by

$$\rho = \frac{R \sinh u}{\cosh u - \cos p}; \; z = \frac{R \sin p}{\cosh u - \cos p}$$

and have the metric coefficients

$$h_\theta = \frac{R \sinh u}{\cosh u - \cos p} \equiv R g_\theta$$

$$h_p = -\,h_u = \frac{R}{\cosh h - \cos p} \equiv Rg$$

where R is the curvature radius of the beam orbit. The minus sign for h_u is required to make E_u point in the same direction as E_ρ when p = 0.

Assuming a time harmonic current density, i_θ, one can write the field vectors as $\mathbf{F} = (F_u\,, F_p\,, jF_\theta)$ exp $j(n\theta\text{-}\omega t)$ with $\omega = vn/R$. Maxwell's equations now take

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the form

$$\frac{\partial}{g\partial p} g_\theta E_\theta + nE_p + vng_\theta H_u = 0$$

$$\frac{\partial}{g\partial u} g_\theta E_\theta - nE_u + vng_\theta H_p = 0$$

$$\frac{\partial}{g\partial u} gE_p + \frac{\partial}{g\partial p} gE_u + vngH_\theta = 0$$

$$\frac{\partial}{g\partial p} g_\theta H_\theta + nH_p - vng_\theta E_u = 0$$

$$\frac{\partial}{g\partial u} g_\theta H_\theta - nH_u - vng_\theta E_p = 0$$

$$\frac{\partial}{g\partial u} gH_p + \frac{\partial}{g\partial p} gH_u - vngE_\theta = i_\theta$$

In the current free region, this set of 6 equations can be reduced to two coupled equations by introducing complex transverse fields, $E_T = E_u + iE_p$ and $H_T = H_u + iH_p$ and the differential operators

$$D \equiv \frac{\partial}{\partial u} + i\frac{\partial}{\partial p};\ D^* \equiv \frac{\partial}{\partial u} - i\frac{\partial}{\partial p}$$

leading to

$$\operatorname{Re} Dg_\theta D^* g_\theta E_\theta - n^2(1-v^2)g^2 E_\theta - vn \operatorname{Re} Dig(g_\theta^2 - 1)H_T = 0$$

and

$$\operatorname{Re} Dg_\theta D^* g_\theta H_\theta - n^2(1-v^2)g^2 H_\theta + vn \operatorname{Re} Dig(g_\theta^2 - 1)E_T = 0$$

together with the expressions for the transverse components, of which only H_p is needed:

$$n(1 - v^2 g_\theta^2)H_p = -\frac{\partial}{g\partial p} g_\theta H_\theta + vg_\theta \frac{\partial}{g\partial u} g_\theta E_\theta$$

3 PERTURBATIVE SOLUTION

For small curvature, i.e. beam tube radius divided by beam orbit radius, $b/R << 1$, one has $u \cup \infty$, and thus $\cosh u \cup \sinh u \cup \exp(u)/2$, resulting in $g_\theta \cup 1$ which implies almost decoupled equations for the azimuthal components. The weak coupling can be conveniently handled by a perturbation method, in which the metric coefficients are asymptotically approximated by

$$g \sim \frac{2}{e^u - 2\eta \cos p};\ g_\theta \sim \frac{e^u}{e^u - 2\eta \cos p}$$

with $\eta = 1$ the perturbation parameter. In order to simplify the solution, a change of the radial coordinate is indicated from u to $x = 2\,(n/\gamma)e^{-u}$ with the filamentary beam at $x = 0$. The beam tube radius and the wall boundary locus can now be approximated by $x_b \approx (n/\gamma)\ (b/R)$ since the resulting off-center beam position is a second order effect.

The differential operator becomes in the new coordinate system

$$D \equiv -x\frac{\partial}{\partial x} + i\frac{\partial}{\partial p}$$

The solution takes a simple and transparent form by restricting the present study to the representative case of a filamentary beam thereby neglecting the space charge effect which decreases with γ^{-2}. The azimuthal current density for a current $I = 2\pi$ is given by $i_\theta = \delta(x - 0)$ (the time harmonic factor is suppressed) and one can write the associated "TM_{01}" - like perturbative solution as follows

$$E_\theta = E_{\theta 0}(x) + \eta E_{\theta 1}(x)\cos p + \eta^2\left(E_{\theta 20}(x) + E_{\theta 2p}(x)\cos 2p\right) + \mathrm{K}$$

$$H_\theta = \eta H_{\theta 1}(x)\sin p + \mathrm{K}$$

$$H_p = H_{p0}(x) + \eta H_{p1}(x)\cos p + \mathrm{K}$$

Note that the longitudinal coupling impedance is determined from $E_{\theta 20}$ only and the $E_{\theta 2p}$ term is not required.

Zeroth order solution. In zeroth order, the fields due to a filamentary beam are those of the TM01 mode in a straight beam tube, with the azimuthal component a solution of $L_0(E_{\theta 0}) = 0$ and the boundary condition $E_{\theta 0}(x_b) = 0$ where L_m represents the modified Bessel function differential equation,

$$L_m(y) \equiv x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - \left(x^2 + m^2\right)y$$

The following expressions are required in finding the first order solution,

$$E_{\theta 0} = \frac{n}{v\gamma^2 R}\left[K_0(x) - I_0(x)K_{0b}/I_{0b}\right] \equiv \frac{n}{v\gamma^2 R}C(x)$$

$$H_{p0} = \frac{n}{\gamma R}\left[K_1(x) + I_1(x)K_{0b}/I_{0b}\right] \equiv -\frac{n}{\gamma R}C'(x)$$

with the shorthand notation $I_{0b} = I_0(x_b)$ and $K_{0b} = K_0(x_b)$.

First Order Solution. The first order field components are obtained from Bessel's equation with forcing term,

$$L_1(E_{\theta 1}) = -3\frac{\gamma x^2}{n}E'_{\theta 0} + 2\frac{v\gamma^2 x^2}{n}\left(2H_{p0} + xH'_{p0}\right)$$

$$L_1(H_{\theta 1}) = 2\frac{v\gamma^3 x^2}{n}E'_{\theta 0}$$

where the prime denotes differentiation with respect to the argument. Together with the boundary conditions $E_{\theta 1}(x_b) = 0$ and $H'_{\theta 1}(x_b) = 0$, one finds after some manipulations

$$E_{\theta 1} = \frac{1}{2v\gamma R}\left\{(1-\gamma^2)\frac{x_b}{I_{0b}I_{1b}}I_1(x) + (1+2\gamma^2)xC(x) - (1-\gamma^2)x^2C'(x)\right\}$$

$$H_{\theta 1} = \frac{\gamma}{R}\left\{\frac{x_b}{I_{0b}\left(x_b I_{0b} - I_{1b}\right)}I_1(x) + xC(x)\right\}$$

The transverse field component required for the second order solution is found from

$$H_{p1} = -v\gamma\left(\frac{1}{vx}H_{\theta 1} + E'_{\theta 1}\right) - \frac{v\gamma^2}{n}\left(E_{\theta 0} - xE'_{\theta 0}\right) - 2\frac{v\gamma^4 x}{n}E'_{\theta 0}$$

Second order solution. The second order solution is obtained from the Bessel function differential equation with forcing term,

$$L_0(E_{\theta 20}) = -\frac{3}{2}\frac{\gamma x}{n}\left(E_{\theta 1} + xE'_{\theta 1}\right) - \frac{3}{2}\frac{\gamma^2 x^2}{n^2}\left(2E_{\theta 0} + xE'_{\theta 0}\right) + \frac{v\gamma^2 x^2}{n}\left(2H_{p1} + xH'_{p1}\right) + \frac{1}{2}\frac{v\gamma^3 x^3}{n^2}\left(7H_{p0} + xH'_{\theta 0}\right)$$

together with the boundary condition $E_{\theta 20}(x_b) = 0$. The solution is somewhat lengthy, but can readily be handled with the aid of a computer program such as MACSYMA. Only the second order azimuthal electric field component contributes to the curbature induced coupling impedance; its expression is

$$E_{\theta 20}(0) = \frac{n}{24vR}\left\{\frac{b^2}{R^2}\frac{1}{I_{0b}^2 I_{1b}\left(I_{0b} - I_{1b}\right)}\left[\gamma^2\left(10I_{1b}^2 + 5I_{0b}I_{1b} - 3I_{0b}^2\right) - 6\left(I_{1b}^2 + 2I_{0b}I_{1b} - I_{0b}^2\right) - \frac{1}{\gamma^2}\left(4I_{1b}^2 - 7I_{0b}I_{1b} + 3I_{0b}^2\right)\right] - \frac{5 - 24\gamma^2 + 16\gamma^4}{n^2}\frac{I_{0b}^2 - 1}{I_{0b}^2}\right\}$$

4 THE COUPLING IMPEDANCE

The curvature-induced coupling impedance is obtained from the azimuthal electric field component by the integral

$$Z_n = -\frac{R}{I}\int_0^{2\pi} E_\theta e^{jn\theta}\, d\theta$$

With the use in this paper of the particular current strength, $I = 2\pi$, the general expression for the couping impedance becomes simply $Z_n = -j\, E_{\theta 20}(0)$. Numerical results can be readily obtained for all mode numbers, provided that double-precision Bessel function routines are used. (As example, results for a RHIC-like machine with $\gamma = 100$ and $b/R = 2\times 10^{-4}$ are shown in Fig. 1.)

Approximate expressions valid at very low and very large mode numbers follow from the general expression:

$$\frac{Z_n}{n} \sim -j\frac{Z_0}{4v}\frac{b^2}{R^2}\left(1 - \frac{7}{8}n^2\frac{b^2}{R^2}\right);\ \left(n < 3\gamma R/b\right)$$

$$\frac{Z_n}{n} \sim j\frac{Z_0}{vn^2}\left(\frac{2}{3}\gamma^4 - \gamma^2 + \frac{5}{24}\right);\ \left(n > 3\gamma R/b\right)$$

$$\sim j\frac{4}{9}\gamma Z_0\frac{n_c}{n^2}$$

thereby providing the mathematical prove, that at sufficiently large mode numbers the coupling impedance of a circular machine approaches the free space value, a fact previously suggested, but only proven for a circular beam between parallel plates. Although the present result was derived for a beam tube with circular cross section, it is expected to be valid for any beam tube geometry.

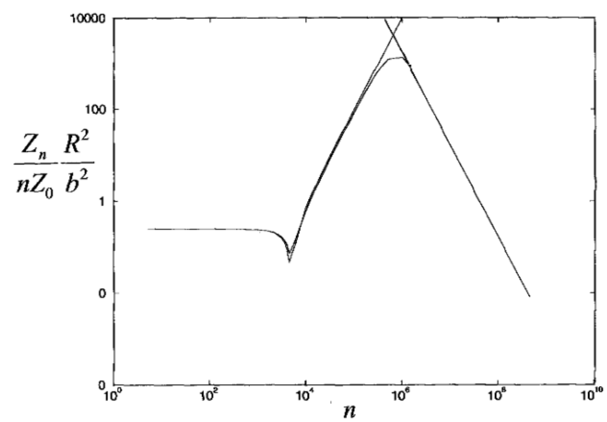

Fig. 1: Exact results and asymptotic approximation for curvature-induced coupling impedance.

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