A STUDY ON MICROWAVE INSTABILITY INDUCED RADIATION[*]

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Abstract

It has been shown in the context of a solvable model that the microwave instability can be described in terms of "coherent states" [1]. Building on this model, we first show that the simplicity of the model arises from the fact that the key integral-differential equation can be reduced to the Karhunen-Loeve equation of the theory of stochastic processes. We present results on the correlation functions of the electric field. In particular, for the second order correlation function, we show that a relation akin to the Hanbury Brown-Twiss correlation holds for the coherent states of the microwave-instability induced radiation. We define an entropy-like quantity and we introduce a Wigner distribution function representation.

1 INTRODUCTION

Recently a new theory of the microwave instability in an electron storage ring was proposed [1]. An analytic solution for the electron distribution function and the electric field was obtained for the model of a bunched electron beam interacting with the model impedance,

$$Z_n = \begin{cases} \overline{Z} & \text{if } n_0 - b \leq n \leq n_0 + b \\ 0 & \text{otherwise} \end{cases}, \qquad (1)$$

with $n > 0$ and $\overline{U} \equiv -in_0\overline{Z}$. The instability in the presence of this impedance can be characterized in terms of a complete set of "periodic, localized coherent states",

$$\sqrt{2\pi(2b+1)}\Gamma_\alpha(\phi) \equiv \sum_{m=-b}^{b} e^{im(\phi-\phi_\alpha)} = \frac{\text{Sin}\left[(2b+1)\left(\dfrac{\phi-\phi_\alpha}{2}\right)\right]}{\text{Sin}\left[\dfrac{\phi-\phi_\alpha}{2}\right]}.$$

$\Gamma_\alpha(\phi)$ is localized near $\phi_\alpha \equiv 2\pi\alpha/(2b+1)$, $\alpha = 0, \pm1, \pm2,$ ..., $\pm b$ with a width in ϕ given by the wakelength, $\Delta\phi = \ell_w \equiv (2b+1)^{-1}$ and a frequency width of $\Delta n = (2b+1)$. The $\Gamma_\alpha(\phi)$ span the space M_b of periodic functions that are bandlimited to $-b \leq n \leq b$, they are orthonormal on [-π, π] and they are complete on M_b,

$$\sum_{\alpha=-b}^{b} \Gamma_\alpha(\phi)\Gamma_\alpha(\phi') = \frac{\sqrt{(2b+1)}}{2\pi}\Gamma_0(\phi-\phi'). \qquad (3)$$

The $\Gamma_\alpha(\phi)$ are simply the basis functions for a periodic version of the classic Shannon sampling theorem [2-3].

2 INITIAL CONDITIONS: SHOT NOISE

The initial electron beam consisting of N particles is

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assumed to be mono-energetic and the electron distribution function in ϕ is given as,

$$F(\phi, t = 0) = \frac{1}{N}\sum_{j=1}^{N}\delta(\phi - \phi_j), \qquad (4)$$

where the individual electrons positions are assumed to be independent and randomly distributed with a weight function $\rho(\phi)$, so-called "shot noise". The average line density is obtained as an ensemble average,

$$\langle F(\phi, 0)\rangle \equiv \int F(\phi_1..\phi_N)\rho(\phi_1)...\rho(\phi_N)d\phi_1...d\phi_N = \rho(\phi). \ (5)$$

Provided that the electron bunch length, $\sigma \gg 2\pi/b \approx \ell_w$, $\rho(\phi)$ is well approximated by its projection into the space M_b, i.e., an expansion in the bandlimited coherent states.

3 APPLICATION OF COHERENT STATES

In [1] the Vlasov-Maxwell equations for the microwave instability were reduced to a Fredholm integral equation for the perturbed electron distribution function $I(\phi, t)$,

$$\ddot{I}(\phi, t) = \kappa\rho(\phi)\int_{-\pi}^{\pi}\Gamma_0(\phi - \phi')\left[e^{in_0(\phi-\phi')}\overline{U} + \text{c.c.}\right]I(\phi', t)d\phi' \ (6)$$

where $\kappa \equiv e\omega_0^2\alpha I_{av}/E_0$, $I_{av} \equiv eN\omega_0/2\pi$ and initial condition $\dot{I}(\phi, 0) = 0$. We define some simplifying notation,

$$I(\phi, t) \equiv e^{in_0\phi}J(\phi, t) + e^{-in_0\phi}J^*(\phi, t) + \breve{I}(\phi, t), \qquad (7)$$

where $J(\phi, t) \in M_b$ is a bandlimited function and $\breve{I}(\phi, t)$ represents that portion of $I(\phi, t) \notin e^{\pm in_0 b} \otimes M_b$. Because $J(\phi, t) \in M_b$ it can be expanded as follows,

$$J(\phi, t) = \sum_\alpha J_\alpha(t)\Gamma_\alpha(\phi), \qquad (8)$$

Using the properties of the $\Gamma_\alpha(\phi)$, equation (6) can be split into a pair of coupled equations for $J(\phi, t)$ and $\breve{I}(\phi, t)$,

$$\ddot{J}(\phi, t) = \kappa\overline{U}\int_{-\pi}^{\pi}\Gamma_0(\phi - \phi')\rho(\phi')J(\phi', t)d\phi', \qquad (9a)$$

$$\ddot{\breve{I}}(\phi, t) = \kappa\sum_\alpha[\rho(\phi) - \rho(\phi_\alpha)]\Gamma_\alpha(\phi)\left[\overline{U}e^{in_0\phi}J_\alpha(t) + \text{c.c.}\right]. \ (9b)$$

We call (9a) the "self consistent closed loop equation" and we call (9b) the "mode coupling equation". Note that (9b) is coupled to (9a), but (9a) is self-contained and can be analyzed alone. We will not consider the mode coupling equation any further here.

4 SELF-CONSISTENT LOOP EQUATION

Substituting in (9a) the coherent state expansions of $J(\phi, t)$, $\rho(\phi)$ and $\Gamma_0(\phi - \phi')$ and simplifying yields,

$$\ddot{J}_\alpha(t) = \kappa\overline{U}\rho_\alpha J_\alpha(t). \qquad (10)$$

Assuming $J_\alpha(t) = J_\alpha(0)e^{-i\Omega_\alpha t}$, where Ω_α is termed the coherent frequency yields the dispersion relation,

$$\Omega_\alpha^2 = -\kappa\overline{U}\rho_\alpha = in_0\kappa\overline{Z}\rho_\alpha. \qquad (11)$$

The coherent states $\Gamma_\alpha(\phi)$ are eigenfunctions of (9a) with a pair of eigenvalues, $\Omega_\alpha^\pm = \sqrt{\pm in_0\kappa\overline{Z}\rho_\alpha}$. To satisfy $\dot{I}(\phi,0) = 0$, the eigensolutions are combined as follows,

$$J(\phi,t) = \sum_{\alpha=-b}^{b} J_\alpha(0)\cos\Omega_\alpha t\,\Gamma_\alpha(\phi). \qquad (12)$$

The physical interpretation is that the spectrum of the initial shot noise within the bandwidth of the impedance is modulated ($\mathrm{Re}\,\Omega_\alpha$) and amplified ($\mathrm{Im}\,\Omega_\alpha$) with a growth rate that depends on the local peak current in the electron beam ρ_α.

5 1st ORDER CORRELATION FUNCTION

The first order correlation function of the bandlimited portion of the current density at two different azimuths ϕ and ϕ' at time t is defined as an ensemble average,

$$G^{(1)}(\phi,\phi',t) \equiv \langle J(\phi,t)J^*(\phi',t)\rangle$$
$$= \sum_{\alpha,\beta=-b}^{b} \langle J_\alpha(0)J_\beta^*(0)\rangle\cos[\Omega_\alpha t]\cos[\Omega_\beta^* t]\Gamma_\alpha(\phi)\Gamma_\beta(\phi'). \qquad (13)$$

To proceed we must relate the correlation function at t = 0 to the initial electron distribution function. We define the "centered particle distribution function", $f(\phi,0) \equiv F(\phi,0) - \rho(\phi)$, this is convenient since $\langle J(\phi,0)\rangle = 0$. It is straightforward to show that,

$$\langle f(\phi,0)f(\phi',0)\rangle = \frac{1}{N}\rho(\phi)\delta(\phi-\phi') - \frac{1}{N}\rho(\phi)\rho(\phi'). \quad (14)$$

It has been shown in [1] that provided that $\lambda \ll \ell_W < \sigma$, the second term in the RHS of equation (14) will not be amplified during the instability and it can be ignored. From above we obtain that,

$$\langle J_\alpha(0)J_\beta^*(0)\rangle = \frac{(eN\omega_0)^2}{N}\rho_\alpha\delta_{\alpha\beta}. \qquad (15)$$

The coefficients in the coherent state expansion of $J(\phi,t)$ are uncorrelated. It is this explicit diagonalization of $G^{(1)}(\phi,\phi',t)$ that makes the coherent states so useful.

Combining the above results, the first order correlation function can be written in a diagonalized form,

$$G^{(1)}(\phi,\phi',t) = \frac{(eN\omega_0)^2}{N}\sum_{\alpha=-b}^{b}\rho_\alpha\,\mathrm{Cos}^2\Omega_\alpha t\,\Gamma_\alpha(\phi)\,\Gamma_\alpha(\phi').(16)$$

This correlation bears a striking resemblance to the mutual coherence function in statistical optics [4]. If there were only one coherent state present, the system could be called "completely coherent"; when several states are present, the system is "partially coherent". In the present problem, for sufficiently long times in the presence of the instability, the coherent state that is centered at the peak

of the initial density distribution, will dominate and the system evolves toward "complete coherence".

6 KARHUNEN-LOÈVE EXPANSION

The correlation function in equation (16), along with the coherent states $\Gamma_\alpha(\phi)$, can be used to write an integral eigenvalue equation as follows,

$$\frac{(Ne\omega_0)^2\rho_\alpha}{N}\Gamma_\alpha(\phi) = \frac{(Ne\omega_0)^2}{N}\int_{-\pi}^{\pi}d\phi'\Gamma_\alpha(\phi')\sum_\beta\rho_\beta\Gamma_\beta(\phi)\Gamma_\beta(\phi')$$
$$= \int_{-\pi}^{\pi}d\phi'\Gamma_\alpha(\phi')\langle J(\phi,0)J^*(\phi',0)\rangle \qquad (17)$$

where $\Gamma_\alpha(\phi)$ is the eigenfunction and $(Ne\omega_0)^2\rho_\alpha/N$ is the eigenvalue. This is precisely the form of the integral eigenvalue equation that is the centerpiece of the Karhunen-Loève expansion for stochastic processes [4]. If equation (10) is multiplied by $\Gamma_\alpha(\phi)$ and summed over α, equation (17) can be used to write the evolution equation for $J(\phi,t)$ as,

$$\ddot{J}(\phi,t) = \kappa\overline{U}\sum_\alpha\int_{-\pi}^{\pi}\rho_\alpha\Gamma_\alpha(\phi)\Gamma_\alpha(\phi')J(\phi',t)d\phi'$$
$$= \frac{N}{(eN\omega_0)^2}\kappa\overline{U}\int_{-\pi}^{\pi}\langle J(\phi,0)J^*(\phi',0)\rangle J(\phi',t)d\phi' \qquad (18)$$

In this case the time dependence complicates the expression but the underlying form is again the integral eigenvalue equation of the Karhunen-Loève expansion where the kernel is simply a symmetric correlation function expanded in terms of the eigenvalues and eigenfunctions [4].

7 WIGNER DISTRIBUTION FUNCTION

The Wigner distribution function can be introduced to yield a quasi-probability density in both the angular position ϕ and its Fourier conjugate n [5-6]. For a stochastic process the Wigner distribution function is defined in terms of the first order correlation function [6],

$$W(\phi,n,t) \equiv \int_{-\pi}^{\pi}G^{(1)}\left(\phi+\frac{\xi}{2},\phi-\frac{\xi}{2},t\right)e^{-in\xi}d\xi. \quad (19)$$

Substituting equation (16), the Wigner function can be written as a sum over the coherent modes,

$$W(\phi,n,t) = \sum_{\alpha=-b}^{b}\rho_\alpha|\mathrm{Cos}(\Omega_\alpha t)|^2 W_\alpha(\phi,n), \qquad (20)$$

where

$$W_\alpha(\phi,n) = \int_{-\pi}^{\pi}\Gamma_\alpha\left(\phi+\frac{\xi}{2}\right)\Gamma_\alpha^*\left(\phi-\frac{\xi}{2}\right)e^{-in\xi}d\xi. \qquad (21)$$

A routine procedure leads to the result,

$$W_\alpha(\phi,n) = \frac{\left(\begin{array}{c}\mathrm{Sin}[(2b-2|n|+1)(\phi-\phi_\alpha)]+\\[2mm]\dfrac{4}{\pi}\sum_{k=0}^{b-1}\dfrac{(-1)^{k-n}(1+2k)}{(1+2k)^2-4n^2}\mathrm{Sin}[(b-k)2(\phi-\phi_\alpha)]\end{array}\right)}{(2b+1)\mathrm{Sin}[(\phi-\phi_\alpha)]}$$

for $-b \le n \le b$ and $W_\alpha(\phi, n) = 0$ otherwise. For the special case of a coasting beam, where all the eigenvalues are the same, all the summations in equation (22) can be done and the Wigner distribution function takes a particularly simple form,

$$W(\phi, n, t) = \begin{cases} \dfrac{1}{2\pi} |Cos(\Omega t)|^2, & -b \le n \le b \\ 0, & n \text{ otherwise} \end{cases} \quad . \quad (23)$$

From the Wigner distribution function we can obtain the marginal distribution functions,

$$\Phi(\phi, t) \equiv \sum_{n=-b}^{b} W(\phi, n, t) = 2\pi \sum_{\alpha=-b}^{b} \rho_\alpha |Cos\Omega_\alpha t|^2 \Gamma_\alpha^2(\phi), \quad (24a)$$

$$\Psi(n, t) \equiv \int_{-\pi}^{\pi} W(\phi, n, t) d\phi = \frac{2\pi}{2b+1} \sum_{\alpha=-b}^{b} \rho_\alpha |Cos\Omega_\alpha t|^2 . \quad (24b)$$

8 RADIATION POWER

In [1] it was shown that $G^{(1)}(\phi, \phi', t)$ can be used to obtain the power in the electric field as follows,

$$\langle P(\phi, t) \rangle = \hat{R} \frac{(eN\omega_0)^2}{N} \sum_{\alpha=-b}^{b} \rho_\alpha Cos^2\Omega_\alpha t \Gamma_\alpha(\phi) \Gamma_\alpha(\phi). \quad (25)$$

If we integrate (25) over all ϕ we find that the total power is the sum over the coherent states, the intensity of each state depends on the initial shot noise, ρ_α, and the growth rate of the instability, $Im\Omega_\alpha$,

$$\langle P(t) \rangle \equiv \sum_{\alpha=-b}^{b} P_\alpha(t) = \hat{R} \frac{(eN\omega_0)^2}{N} \sum_{\alpha=-b}^{b} \rho_\alpha |cos[\Omega_\alpha t]|^2 \quad (26)$$

We can also define the power as a function of harmonic number in terms of one of the marginal distributions of the Wigner distribution function,

$$\langle P(n, t) \rangle = \hat{R} \frac{(eN\omega_0)^2}{N} \frac{1}{(2b+1)} \sum_{\alpha=-b}^{b} \rho_\alpha |cos[\Omega_\alpha t]|^2$$
$$= \hat{R} \frac{(eN\omega_0)^2}{N} \frac{\Psi(n, t)}{2\pi} \quad (27)$$

for $-b \le n \le b$ and zero otherwise. The fact that $\langle P(n, t) \rangle$ is independent of "n" is due to the fact that Z has been assumed to be constant. If the expression in (27) is summed over "n" we again obtain the result (26) for the total power.

9 ENTROPY

Paralleling the analysis for coherent light we can define an entropy [7],

$$H(t) \equiv -\sum_\alpha \Theta_\alpha(t) Log\Theta_\alpha(t) \quad (28)$$

where

$$\Theta_\alpha(t) \equiv \rho_\alpha |Cos[\Omega_\alpha t]|^2 / \sum_\alpha \rho_\alpha |Cos\Omega_\alpha t|^2 , \quad (29)$$

$\Theta_\alpha(t)$ is the relative probability that a given coherent mode is excited. It is straightforward to prove that $H_{min}(0) = 0$ when $\Theta_\alpha(0) = 1$, $\Theta_\beta(0) = 0, \alpha \neq \beta$, &

$H_{max}(0) = Log(2b+1)$ when $\rho_\alpha = [2\pi(2b+1)]^{-1/2}$ for all α. Since the entropy is the logarithm of the number of coherent modes excited on the beam it is also a measure of "coherence", lower entropy implies greater coherence. For a Gaussian electron beam, where many modes are initially excited; the entropy will decrease toward zero with time as the mode with the largest growth rate dominates.

10 2nd ORDER CORRELATION FUNCTION

The second order correlation function is defined as,

$$G^{(2)}(\phi, \phi', \phi'', \phi''', t) \equiv \langle J(\phi, t) J^*(\phi', t) J(\phi'', t) J^*(\phi''', t) \rangle . \quad (30)$$

After considerable manipulation in can be shown that the second order correlation function can be written in solely in terms of the first order correlation function,

$$G^{(2)}(\phi, \phi', \phi'', \phi''', t)$$
$$= [G^{(1)}(\phi, \phi', t) G^{(1)}(\phi'', \phi''', t) + G^{(1)}(\phi, \phi''', t) G^{(1)}(\phi', \phi'', t)] \quad (31)$$

Using this key result we can derive a Hanbury-Brown Twiss-like relation [4,8] between the power in individual coherent states,

$$\frac{\langle P_\alpha(t) P_\beta(t) \rangle}{\langle P_\alpha(t) \rangle \langle P_\beta(t) \rangle} = 1 + \delta_{\alpha\beta} . \quad (32)$$

This implies there is an enhanced correlation of light in a particular coherent state, a characteristic of so-called "chaotic light". However there is no such correlation between two different coherent states.

It can be shown that the total radiation power satisfies,

$$\frac{\langle P(t)^2 \rangle}{\langle P(t) \rangle^2} = 1 + \frac{\sum_\alpha \langle P_\alpha(t) \rangle^2}{\langle P(t) \rangle^2} . \quad (33)$$

For the specific case of a "flat electron beam" of bunch length $\ell_b = 2\pi \frac{2a+1}{2b+1}$ equation (33) simplifies to,

$$\frac{\langle P(t)^2 \rangle}{\langle P(t) \rangle^2} = 1 + \frac{1}{2a+1} \quad (34).$$

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