UNIFIED TREATMENT OF COLLECTIVE INSTABILITIES AND NONLINEAR BEAM DYNAMICS

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Abstract

Nonlinear dynamics deals with parametric resonances and diffusion, which are usually beam-intensity independent and rely on a particle Hamiltonian. Collective instabilities deal with beam coherent motion, where the Vlasov equation is frequently used in conjunction with a beam-intensity dependent Hamiltonian. We address the questions: Are the two descriptions the same? Are collective instabilities the results of encountering parametric resonances whose driving force is intensity dependent? The space-charge dominated beam governed by the Kapchinskij-Vladimirskij (K-V) envelope equation [1] is used as an example.

1 INTRODUCTION

Traditionally, the thresholds of collective instabilities are obtained by solving the Vlasov equation, the dynamics of which comes from a wakefield-dependent Hamiltonian. The unperturbed beam distribution is computed using the unperturbed part of the Hamiltonian H_0, which takes care of the mean field and potential-well distortion. The perturbation distribution is obtained by solving the Vlasov equation that involves the perturbation Hamiltonian ΔH_1. The Vlasov equation is often linearized so that the modes of collective motion can be described by a set of orthonormal eigenfunctions and the corresponding complex eigenvalues give the initial growth rates. ΔH_1 may have a time-independent component, for example, the part involving the nonlinear magnetic fields, that gives rise to the dynamical aperture limitation. It may also have a time-dependent component, which includes the effects of wakefields and produces coherent motion of beam particles. The harmonic content of the wakefields depends on the structure of accelerator components. If one of the resonant frequencies of the wakefields is equal to a fractional multiple of the unperturbed tune of H_0, a resonance is encountered and coherent particle motion is introduced. This may result in a runaway situation such that collective instability is induced.

Experimental measurements indicate that a small time dependent perturbation can create resonance islands in the longitudinal or transverse phase space and profoundly change the bunch structure [2]. For example, a modulating transverse dipole field close to the synchrotron frequency can split up a bunch into beamlets. Although these phenomena are driven by beam-intensity independent sources, they can also be driven by the space-charge force and/or the wakefields of the beam which are intensity dependent. Once perturbed, the new bunch structure can further enhance the wakefields inducing even more perturbation to the circulating beam. Experimental observation of hysteresis in collective beam instabilities seems to indicate that resonance islands have been generated by the wakefields.

For example, the Keil-Schnell criterion [3] of longitudinal microwave instability can be derived from the concept of bunching buckets, or islands, created by the perturbing wakefields. Particles in the beam will execute *synchrotron* motion inside these buckets leading to growth in the momentum spread of the beam. In fact, the collective growth rate is exactly equal to the angular synchrotron frequency inside these buckets. If the momentum spread of the beam is much larger than the bucket height, only a small fraction of the particles in the beam will be affected and collective instabilities will not occur. This mechanism has been called Landau damping.

As a result, we believe that the collective instabilities of a beam can also be tackled from a particle-beam nonlinear-dynamics approach, with collective instabilities occurring when the beam particles are either trapped in resonance islands or diffuse away from the beam core because of the existence of a sea of chaos. The advantage of the particle-beam nonlinear dynamics approach is its ability to understand the hysteresis effects and to calculate the beam distribution beyond the threshold condition. Such a procedure may be able to unify our understanding of collective instabilities and nonlinear beam dynamics. Here, the stability issues of a space-charge dominated beam in a uniformly focusing channel are considered as an example [4].

2 COLLECTIVE-MOTION APPROACH

Gluckstern *et. al.* [5] have studied the collective beam stabilities of a space-charge dominated K-V beam in a uniformly focusing channel. They showed that the (1,0) mode is stable for any amount of envelope mismatch and tune depression η. The (2,0) mode becomes unstable at zero mismatch when $\eta < 1/\sqrt{17} = 0.2435$ and also when the mismatch is large. This is plotted in Fig. 1 with the stable region enclosed by the red solid curve. The stability regions of the (3,0) and (4,0) modes, enclosed by the blue dashes and the magenta dot-dashes, respectively, are also shown. These latter two modes become unstable at zero mismatch when the tune depressions are less than 0.3859 and 0.3985, respectively. They found that the modes become more unstable as the number of radial nodes increases. Among all the azimuthals, they also noticed that the azimuthally symmetric modes (ℓ,0) are the most unstable.

3 PARTICLE-BEAM APPROACH

We want to investigate whether the instability regions in Fig. 1 can be explained by nonlinear parametric resonances. The particle Hamiltonian describing an azimuthally symmetric oscillating beam core of radius R is [6]

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$$H_p = \frac{1}{4\pi}p_y^2 + \frac{\mu^2}{4\pi}y^2 - \frac{2\mu\kappa}{4\pi R^2}\,y^2\,\Theta(R-|y|) - \frac{2\mu\kappa}{4\pi}\left(1+2\ln\frac{|y|}{R}\right)\Theta(|y|-R)\,. \qquad (1)$$

where y and p_y are the particle's transverse coordinate and canonical momentum, $\mu/(2\pi)$ the *unperturbed* particle's betatron tune, and κ the normalized space-charge perveance, which is related to the tune depression by $\eta = \sqrt{1+\kappa^2}-\kappa$. Here, only the situation of zero angular momentum is discussed [4]. For a weakly mismatched beam, the envelope radius can be written as $R=R_0+\Delta R\cos Q_e\theta$, where Q_e is the envelope tune and θ the 'time'. The particle Hamiltonian can be expanded in terms of the equilibrium envelope radius R_0, resulting in $H_p=H_{p0}+\Delta H_p$, where the unperturbed Hamiltonian H_{p0} is the same as H_p with R replaced by R_0. Thus, for a matched beam, $\Delta H_p = 0$.

4 PARAMETRIC RESONANCES

For a mismatched beam, particle motion is modulated by the oscillating beam envelope. The perturbation Hamiltonian ΔH_p, obtained from Taylor's expansion, can be expanded as a Fourier series in the action-angle variables [6]. Parametric resonances occur when the phase is stationary. Focusing on the n:m resonance, we perform a canonical transformation to the resonance rotating frame (I_p, ϕ_p):

$$\langle H_p\rangle = E_p(I_p) - \frac{m}{n}Q_e I_p + h_{nm}(I_p)\cos n\phi_p\,, \qquad (2)$$

with the effective κ-dependent resonance strength given by

$$h_{nm} = \frac{(m+1)M^m\mu\kappa}{2\pi R_0^2}\,|G_{nm}(I_p)|\,, \qquad (3)$$

where $M = 1 - R_{\rm min}/R_0$ is the envelope mismatch. The n stable and unstable fixed points can be found easily. Because particles are affected only by resonances when they are just outside the envelope core, their tunes are essentially the tune inside the beam envelope. At zero mismatch, the threshold for the n:m resonance can therefore be derived by equating the ratio of particle to envelope tunes to m/n, i.e.,

$$\kappa \ge \frac{1}{2\sqrt{2}}\left[\left(\frac{n}{m}\right)^2-4\right]\left[\left(\frac{n}{m}\right)^2-2\right]^{-1/2}. \qquad (4)$$

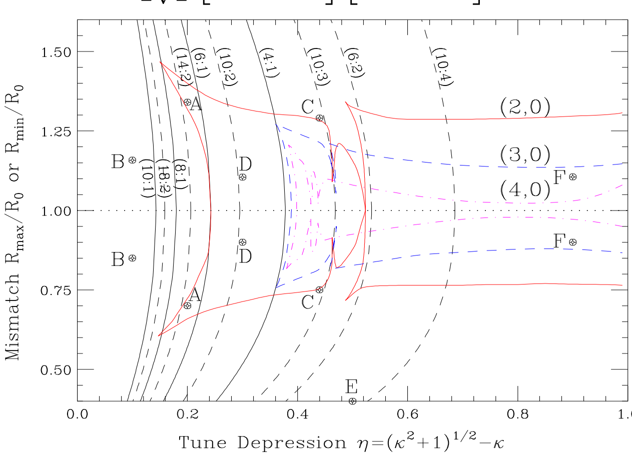

Figure 1: Beam stability versus particle tune depression and envelope mismatch: stability region for Gluckstern's (2,0) mode enclosed by red solid curve, the (3,0) and (4,0) modes by blue dashes curve and magenta dot-dashes. Overlaid are first-order resonances shown as solid and second- and higher-order resonances as dashes.

In particular, for the 6:1 resonance, $\kappa\ge 8/\sqrt{17}=1.9403$, or tune depression $\eta\le 1/\sqrt{17} = 0.2425$, which agrees with Gluckstern's instability threshold for the (2,0) excitation.

Trackings have been performed for particles outside the envelope core using the 4th-order symplectic integrator [7]. The Poincaré surface of section are shown in Plots A, B, C, D, E, F of Fig. 2 corresponds to Points A, B, C, D, E, F in Fig. 1. The innermost torus is the beam envelope. The sections are taken every envelope oscillation period when the envelope radius is at a minimum. In Plot A, with $(\eta, M) = (0.20, 0.30)$, particles that diffuse outside the beam envelope, will encounter the 6:1 resonance, which is bounded by a very thin layer of tori. This region is therefore on the edge of instability. However, the last good torus will be broken if η is further decreased, which corresponds to Plot B, a close-up plot with $(\eta, M) = (0.10, 0.15)$. Particles that diffuse outward from the beam core will wander easily towards the 2:1 resonance along its separatrix. This region, where $\eta \lesssim 0.2$, is therefore very unstable. This explains the front stability boundary of the (2,0) mode of Gluckstern, *et. al*. Particles in Plot C with $(\eta, M) = (0.44, 0.25)$ see many parametric resonances, first 10:3, then 6:2, 8:3, 10:4 and after that a chaotic layer going towards the 2:1 resonance. These resonances are separated by thin layers of good tori. This region is on the edge of instability. Plot D with $(\eta, M) = (0.30, 0.10)$ shows the 6:2 resonance well separated from the 10:4 resonance with a wide area of good tori. Note that the 2:1 unstable fixed points and separatrices are not chaotic at all. This region will be very stable. Plot E, with $(\eta, M) = (0.50, 0.60)$, is at very large mismatch although the tune depression is moderate. The 2:1 unstable fixed points and separatrices are very chaotic, and are very close to the beam core. Thus particles can easily diffuse towards the 2:1 resonance, making this region unstable. Finally, Plot F, with $(\eta, M) = (0.90, 0.10)$, is with small space charge and small mismatch. The beam envelope is surrounded by good tori far away from the 2:1 separatrices. This region is very stable.

Since the 4:1 resonance is a strong one, its locus explains the front stability boundaries of Gluckstern's (3,0) and (4,0) modes also. The deep fissures of the (2,0) mode near $\eta = 4.7$ and 5.3 are probably the result of encountering the 10:3 and 6:2 parametric resonances. The width of the fissures should be related to the width of the resonance islands, which can be computed in the standard way. In general, a first-order resonance island, like the 4:1, is much wider than a higher-order resonance island, like the 6:1.

We tried very hard to examine the region between the 4:1 and 10:3 resonances with a moderate amount of mismatch. We found this region very stable unless it is close to the 10:3 resonance. We could not, however, reproduce the slits that appear in the (4,0) mode of Gluckstern, *et. al*.

5 CONCLUSIONS

We have now an interpretation of the collective instabilities in the plane of envelope mismatch and tune depression through the particle-beam nonlinear-dynamics approach.

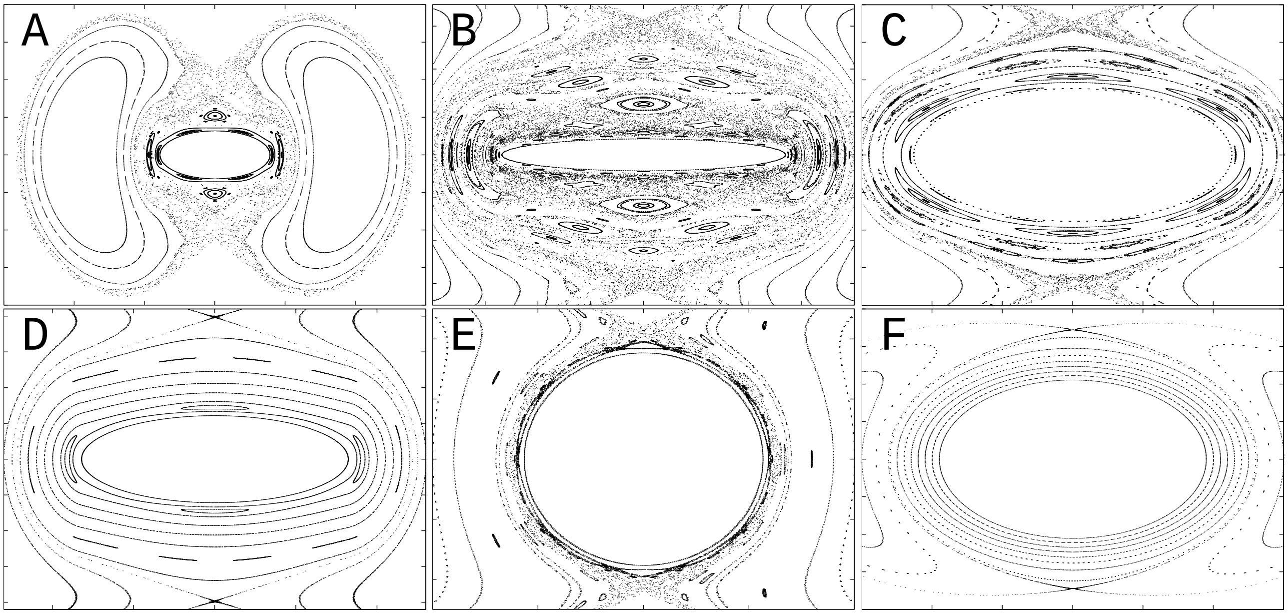

Figure 2: Poincaré surface of section in particle phase space (y, p). Plot A is with $(\eta, M) = (0.20, 0.30)$, Plot B (0.10, 0.15), Plot C (0.44, 0.25), Plot D (0.30, 0.10), Plot E (0.50, 0.60), Plot F (0.90, 0.10), corresponding, respectively, to Points A, B, C, D, E, F in Fig. 1. The last 5 are close-up plots, showing only up to the unstable fixed points and internal separatrices of the 2:1 resonance.

Because of the existence of noises of all types in the accelerators and the K-V equation is far from realistic, some particles will diffuse away from the K-V distribution. Although these particles may encounter parametric resonances once outside the beam core, an equilibrium will be reached if these resonances are bounded by invariant tori. It may happen that the island chains outside the beam envelope are so close together that they overlap to form a chaotic sea. When the last invariant torus breaks up, particles leaking out from the core diffuse towards the 2:1 resonance, which is usually much farther away from the beam envelope, to form beam halos. As particles escape from the beam envelope, the beam intensity inside the envelope becomes smaller and the equilibrium radius of the beam core shrinks. Thus more particles will find themselves outside the envelope. As this process continues because no equilibrium can be reached, the beam eventually becomes unstable.

So far, we have been able to explain the results of Gluckstern, *et. al* qualitatively. However, there are differences quantitatively. To the lowest order, the Vlasov equation studied by Gluckstern, *et. al.* does involve the perturbation force induced by the *perturbation* distribution via the Poisson's equation. In our nonlinear-dynamics approach, the particle that escapes from the beam envelope core, always sees the Coulomb force of the *entire unperturbed beam core*, independent of any variation of the core distribution due to the leakage of particles. This is mainly due to the fact that we have been treating the envelope Hamiltonian and the particle Hamiltonian separately. This leads to a dependency of the particle equation of motion on the envelope radius, but not the dependency of the equation of motion of the envelope radius on the particle motion. This is a shortcoming in our approach, which we need to improve. We believe that this is also the reason why we have not been able to compute the growth rates of the instabilities.

It is possible that many collective instabilities can be explained by the particle-beam nonlinear dynamics approach. The wakefields of the beam interacting with the particle distribution produce parametric resonances and chaotic regions. Collective instabilities will be the result of particles trapped inside these resonance islands. The perturbed bunch structure further enhances the wakefields to induce these collective particle instabilities.

6 REFERENCES

[1] I.M. Kapchinskij and V.V. Vladimirskij, Proc. Int. Conf. on High Energy Accelerators, CERN, Geneva, 1959, p.274.

[2] D.D. Caussyn, *et. al.*, Phys. Rev. **A46**, 7942 (1992); M. Ellison, *et. al.*, Phys. Rev. Lett. **70**, 591 (1993); M. Syphers, *et. al.*, Phys. Rev. Lett. **71**, 720 (1993); D. Li, *et. al.*, Phys. Rev. **E48**, R1638 (1993); DD. Li, *et. al.*, Phys. Rev. **E48**, 3 (1993); H. Huang, *et. al.*, Phys. Rev. **E48**, 4678 (1993); Y. Wang, *et. al.*, Phys. Rev. **E49**, 1610 (1994); Y. Wang, *et. al.*, Phys. Rev. **E49**, 5697 (1994); S.Y. Lee, *et. al.*, Phys. Rev˙ **E49**, 5717 (1994); M. Ellison, *et. al.*, Phys. Rev. **E50**, 4051 (1994); L.Y. Liu, *et. al.*, Phys. Rev. **E50**, R3344 (1994); D. Li, *et. al.*, Nucl Inst. Meth. **A364**, 205 (1995).

[3] E. Keil and W. Schnell, CERN Report SI/BR/72-5, 1972; D. Boussard, CERN Report Lab II/RF/Int./75-2, 1975.

[4] K.Y. Ng and S.Y. Lee, "Paricle-Beam Approach to Collective Instabilities—Application to Space-Charge Dominated Beams," Proc. 16th Advaced ICFA Beam Dynamics Workshop on Nonlinear and Collective Phenomena in Beam Physics, Arcidosso, Sep. 1-5, 1998.

[5] R.L. Gluckstern, W-H. Cheng, and H. Ye, Phys. Rev. Lett. **75**, 2835 (1995); R.L. Gluckstern, W-H. Cheng, S.S. Kurennoy, and H. Ye, Phys. Rev. **E54**, 6788 (1996).

[6] S.Y. Lee and A. Riabko, Phys. Rev. **E51**, 1609 (1995).

[7] E. Forest and M. Berz, LBL Report LBL-25609, ESG-46, 1989.