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TRANSVERSE NONLINEAR FOCUSING OF NONSTATIONARY SPACE CHARGE DOMINATED BEAMS*

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Abstract

The general analytical and numerical scheme to calculate the parameters of high intensity beam being transported is considered. Nonlinearities of external fields and space charge are taken into account. The matching conditions for a beam and focusing system aren't required. Lie algebraic technique was applied to derive the dynamic and the field equations selfconsistently. The distribution function and the macroscopic parameters of a beam at any transport channel cross-section were calculated in the framework of the Heisenberg picture in statistical mechanics. The computer code was carried out and verified. Test results are represented.

1 INTRODUCTION

To calculate the main dynamic parameters of the continuous relativistic high-current beam, being focused, at any cross-section of the transport channel one should operate with altering distribution function of transverse coordinates and momenta of charged particles. For chosen temporal and spatial scales interaction between particles inheres in collective behavior, charged plasma assumes to be collisionless. Therefore, firstly, one may consider a beam, submitted to the electromagnetic fields of focusing elements, as the Hamiltonian system. Secondly, to calculate the macroscopic parameters of a beam we may evaluate one particle distribution function $g(x, y, p_x, p_y; z)$ that satisfies the Vlasov equation.

For the realistic transport channels operations with a small parameter and linearized selfconsistent equations are not valid. One should implement nonpertubative methods to step forward. In [1] to apply Lie algebraic techniques were proposed and stationary case was examined. Nonstationary focusing was considered in [2] on the basis of the quasi-stationary plasma model by the algebraic methods. In this paper the general solution of nonlinear focusing of nonstationary space charge dominated beam is discussed.

2 TRANSFER MAP CALCULATION

Let's consider 4D-phase space of transverse canonical conjugated coordinates and momenta of the continuous charged particle beam. The actions of the transfer map M, which brings about the symplectic manifold automorphisms, on the phase variables vector $\xi(x, y, P_x, P_y)$ and on the dynamic function $u(\xi)$ are defined as $\xi(z) = \mathrm{M}\xi$ and $u(\xi, z) = \mathrm{M}u(\xi)$. An independent variable z is the coordinate along the reference trajectory. It is essentially, that the transfer map implementation allows operating with one-particle Hamiltonian. If the initial coordinates and momenta values of an arbitrary particle are known at $z = 0$, i.e. at the start point of the transport channel, using the operator M one may calculate them at any z, i.e. at any cross-section of the transport channel.

One-particle Hamiltonian $H(\xi, z)$, that governing the continuous beam transverse dynamics, may be expressed with sufficient accuracy as a finite sum of m-forms. This is legitimate provided that the transverse energy of an arbitrary particle is considerably less then its oriented motion energy. The number of the m-forms depends on the accuracy required. Without the loss of generality we will be concerned with

$$H(\xi, z) = H_2 + H_3 + H_4 \quad . \tag{1}$$

Here $H_2(\xi, z) = 1/2 \cdot S_{ij}(z)\xi_i\xi_j$,

$$H_3(\xi, z) = T_{ijk}(z)\xi_i\xi_j\xi_k ,$$

$$H_4(\xi, z) = L_{ijkl}(z)\xi_i\xi_j\xi_k\xi_l .$$

Summation over repeated indexes is implied. Matrixes $S_{ij}, T_{ijk}, L_{ijkl}$ are symmetric for any pair of indexes and depend on z.

Taking into consideration (1) it is reasonable to find out the transfer map M structure according to the Dragt-Finn factorization theorem [3]:

$$\mathrm{M} = \exp(:f_4:)\exp(:f_3:)\exp(:f_2:) ,$$

where $:f_m:$ is the Lie operator associated with the homogeneous polynomial of degree m.

The dynamic equations for the transfer map factors were received in [3]. After algebraic manipulation they are casted into the matrix form:

$$\begin{aligned} \dot{M}_{ij} &= J_{ia}S_{ab}M_{bj} \quad , \\ \dot{F}_{ijk} &= -T_{abc}M_{ai}M_{bj}M_{ck} \quad , \\ \dot{G}_{ijkl} &= -L_{abcd}M_{ai}M_{bj}M_{ck}M_{dl} - \\ &\quad -9/2 \cdot F_{dij}T_{abc}M_{ar}M_{bk}M_{cl}J_{dr} \quad . \end{aligned} \tag{2}$$

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Here $M(z)=\exp(:f_2:)$, $f_3=F_{ijk}(z)\xi_i\xi_j\xi_k$, $f_4=G_{ijkl}(z)\xi_i\xi_j\xi_k\xi_l$. Matrixes $M_{ij}, F_{ijk}, G_{ijkl}$ are also symmetric for any pair of indexes and essentially dependent on z. The unitary symmetric and skew-symmetric matrixes are denoted as I_{ij} and J_{ij}. The differential equations (2) have to be solved subject to $M_{ij}(z=0)=I_{ij}$, $F_{ijk}(z=0)=0$, $G_{ijkl}(z=0)=0$.

If the explicit form of the transfer map M is known, one may calculate changed in time values of a phase variable vector ξ according to

$$\xi(z)=M\xi+(:f_3:)M\xi+ \\ +(\frac{:f_3:^2}{2})M\xi+(:f_4:)M\xi \quad , \qquad (3)$$

a dynamic function $u(\xi)$ - according to

$$u(\xi(z))=u(M\xi)+u((:f_3:)M\xi)+ \\ +u\left((\frac{:f_3:^2}{2})M\xi\right)+u((:f_4:)M\xi) \quad , \qquad (4)$$

a beam macroscopic parameter $U(z)$ - according to

$$U(z)=\int dx\int dy\int dp_x\int dp_y \times \\ u(\xi(z))\mathrm{g}(x,y,p_x,p_y) \quad . \qquad (5)$$

The last formula assumes that averaging the microscopic dynamic function over an ensemble of particles implies the Heisenberg picture in statistical mechanics.

To evaluate the dynamic equations (2) one should know the z-dependence of the S, T, L matrixes. A nontrivial question is to evaluate the electromagnetic terms in the Hamiltonian (1). The electromagnetic forces acting on a beam particle are due to the external fields of focusing elements and to the interaction of a particle with its environment.

3 FOCUSING SYSTEM POTENTIALS

If in particular the magnetic focusing system is used to transport high-current relativistic beam, we have $\varphi^{\text{field}}(x,y;z)=0$. As for the vector $\mathbf{A}^{\text{field}}(x,y;z)$, its components are calculated analytically in 3 stages for each focusing element.

1) Solve the Laplace equation for the magnetostatic potential $U(x,y,z)$ taking into account the boundary conditions implied by the magnetic field symmetry.

2) Calculate the components of the magnetic induction $\mathbf{B}(x,y,z)$ from $\mathbf{B}(x,y,z)=\mathrm{grad}U(x,y,z)$.

3) Obtain the vector potential $\mathbf{A}(x,y,z)$ projections from $\mathrm{rot}\mathbf{A}(x,y,z)=\mathbf{B}(x,y,z)$.

4 SPACE CHARGE POTENTIALS

To calculate $\varphi^{\text{beam}}(x,y;z)$ one should solve the Poisson equation. Its solution at an arbitrary point (x_0,y_0) of the beam cross-section at some z is

$$\varphi^{beam}(x_0,y_0;z)=-\frac{1}{\varepsilon_0}\frac{I}{v_0}\int dx'\int dy'\int dp'_x\int dp'_y \times \\ G(x_0,y_0,x',y')g(x',y',p'_x,p'_y) \quad , \qquad (6)$$

where I is a beam current, v_0 is a reference particle velocity, ε_0 is a dielectric permittivity of free space. And $G(x_0,y_0,x',y')$ denotes the Green's function.

It should be noted that the limits of integration over x',y',p'_x,p'_y are unknown. In general case one must substitute the initial variables x,y,p_x,p_y for transformed variables x',y',p'_x,p'_y in the integral, that should be of the form (5). As a result we conclude, that integration involves the inverse transfer map M^{-1}.

But if the initial distribution $g(x,y,p_x,p_y)$ is the Gaussian, transformed distribution $g(x',y',p'_x,p'_y)$ will be the Gaussian too. Moreover, if variables in $g(x,y,p_x,p_y)$ are not coupled, the same is valid for variables in $g(x',y',p'_x,p'_y)$. And for that distribution we may establish the limits of integration in (6) through the values $\sigma'_x=\sqrt{<x'^2>}, \sigma'_y=\sqrt{<y'^2>}$ and $\lambda'_x=\sqrt{<p'^2_x>}, \lambda'_y=\sqrt{<p'^2_y>}$ according to the "3 sigma" rule.

After integrating over momenta (6) takes the form

$$\varphi^{beam}(x_0,y_0;z)=-\frac{1}{\varepsilon_0}\frac{I}{v_0}\int_{-3\sigma'_x}^{3\sigma'_x}dx'\int_{-3\sigma'_y}^{3\sigma'_y}dy' \quad \times \\ \times\frac{G(x_0,y_0,x',y')}{(2\pi)\sigma'_x\sigma'_y}\exp\left(-\frac{1}{2}\left[\frac{x'^2}{\sigma'^2_x}+\frac{y'^2}{\sigma'^2_y}\right]\right) \quad (7)$$

Variable substitution $x'\to x'/\sigma'_x$, $y'\to y'/\sigma'_y$ allows choosing $G(x_0,y_0,x',y')$ as the Green's function of the inner Dirichlet problem for a circle [4].

When the explicit form of the transfer map M is known, we integrate (7) numerically at the knots of a spatial net, which covers the cross-section of a beam.

Hence, to calculate the transfer map factors, firstly, we should represent $\varphi^{\text{beam}}(x',y';z)$ as a finite sum of m-forms according to (1). Secondly, we should establish the z-dependence of the coefficients of the m-forms.

To satisfy the first requirement we consider $\varphi^{\text{beam}}(x',y';z)$ as a function of 2 variables x',y' and

1 parameter z. It is substantial, that the function is defined within the circle. Using the Chebyshev polynomials as a complete set of orthogonal functions we decompose $\varphi^{\text{beam}}(x', y'; z)$ on the polynomials of x', y' up to the 4-th degree. Coefficients $C_{ij}(z)$ of an approximation are calculated by the least squares method. Due to the elliptic symmetry we evaluate only 5 of them. As the result should be expressed in variables of (7), after the inverse substitution $x' \to \sigma'_x \cdot x'$, $y' \to \sigma'_y \cdot y'$ we have

$$\varphi^{beam}(x', y'; z) = C_{20}(z)x'^2 + C_{02}(z)y'^2 + \\ + C_{40}(z)x'^4 + C_{04}(z)y'^4 + C_{22}(z)x'^2 y'^2 \quad .$$

To satisfy the second requirement we consider $\varphi^{\text{beam}}(x', y'; z)$ as a function of 1 variable z and 2 parameters x', y'. It is substantial, that any $C_{ij}(z)$ is a monotonous function within some focusing element. So one may construct the empiric formula with 2 parameters for each $C_{ij}(z)$ within each focusing element. We use the modified method of averages to establish the type of a formula and compute its parameters.

To calculate $\mathbf{A}^{\text{beam}}(x', y'; z)$ we solve the vector Poisson equation in the same manner. If there is no a shift of the beam centroid, one may use $A_x = A_y = 0$, $A_z{}^{\text{beam}}(x', y'; z) = \dfrac{v_0}{c^2}\varphi^{\text{beam}}(x', y'; z)$. In general case, the decomposition of functions and the construction of formulas lead to

$$\mathbf{A}^{\text{beam}}(x', y'; z) = \mathbf{D}_{20}(z)x'^2 + \mathbf{D}_{02}(z)y'^2 + \\ + \mathbf{D}_{40}(z)x'^4 + \mathbf{D}_{04}(z)y'^4 + \mathbf{D}_{22}(z)x'^2 y'^2 \quad .$$

5 FRINGE QUADRUPOLES FOCUSING

As an example we consider nonlinear focusing of an electron nonstationary space charge dominated Gaussian beam in a fringe magnetic quadrupole channel. The algebraic approach discussed above was implemented to the computer code LIE_HEI written in Fortran-90.

Let a beam current is $I = 100$ A, a reference particle energy is $E_0 = 1$ MeV, initial centroid parameters are $\bar{x}(0) = 0.375\times10^{-10}$ m, $\bar{y}(0) = -0.575\times10^{-9}$ m, initial sizes are $\tilde{x}(0) = 0.25\times10^{-2}$ m, $\tilde{y}(0) = 0.25\times10^{-2}$ m, initial divergences are of 1%.

Let a quadrupole channel is of the total length 1.25 m and consists of 3 lenses with lengths 0.375, 0.5, 0.375 m respectively. Each lens has the same values of the gradient $g = 0.025$ Tl/m and its second derivative $g'' = 0.001$ Tl/m^3.

In that case of small nonlinearities the moments approach may be used to treat the example [4].

Figures 1 and 2 depict the transverse beam sizes and the beam centroid parameters respectively as functions of z in SI units. The solid curves concern the moments method [4] and the dashed ones concern the algebraic approach. It is clear, that results of different methods are in complete agreement with each other.

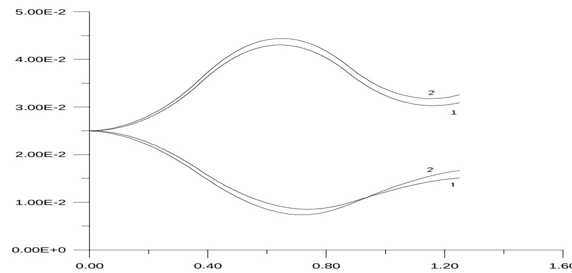

Figure 1: Transverse beam sizes variations.

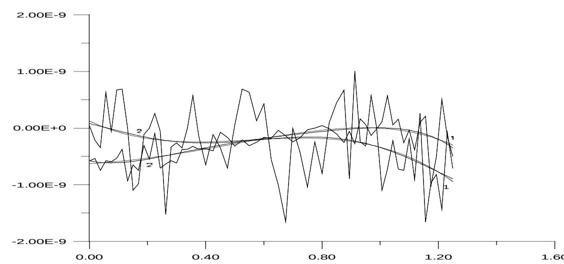

Figure 2: Beam centroid parameters variations.

6 CONCLUSION

The analytical approach to solve the problem of nonstationary nonlinear focusing of a high-current beam was developed. It uses the most general equations that govern a beam dynamics. And it means that the method may have various applications in charged particle beam physics and accelerator science.

Also it should be noted, that using in particular the Heisenberg picture allows to solve the dynamic equations and calculate the beam parameters, including its emittance and brightness, in the same manner and without a concern about the distribution evaluation.

7 REFERENCES

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