ON SELF-CONSISTENT β-FUNCTIONS OF COLLIDING BUNCHES

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Abstract

The flip-flop effect with the linearized beam-beam force is formulated through self-consistent β-functions and equilibrium emittances which are both affected by collision. We give the results of two models of emittance dependence. The effect of finite bunch length is also discussed.

1 INTRODUCTION

From many observations of the beam-beam effects on existing e^+e^- colliders, it is known that under some conditions the sizes of opposing bunches become very different. This phenomenon is called the flip-flop effect. Such a state is not stable and the bunches may exchange their sizes. The flip-flop effect leads to reduction of the luminosity, because of the difference in bunch sizes resulting in reduction of the effective interaction area.

The problem is greatly simplified by *linearization* of the beam-beam force; it has been studied in terms of evolution of the 2nd moments of the beam distribution, involving the radiation effects: damping and quantum excitation [1, 2, 3].

Another way to understand this problem is formulation in terms of self-consistent dynamic β-functions of colliding beams at the interaction point(IP) [4]. The equilibrium emittances of the bunches are affected by the linear part of the beam-beam force: action of the opposing bunch is roughly equivalent to insertion of a (thin) lens modifying the arc lattice [5]. So, a correct account for these *dynamic* emittance variations should be done in a self-consistent way.

This paper gives results of the self-consistent model for round colliding beams, and calculation of the equilibrium radiation emittance with the thin lens insertion. We also discuss the simple model representing the bunch length effect.

2 SELF-CONSISTENT β-FUNCTIONS

Consider a collider lattice with one IP and the betatron phase advance on the arc $\mu_0 = 2\pi\nu$. We can get the resulting matrix of one revolution $M = M_0 \cdot F$ multiplying the arc matrix M_0 by the thin lens matrix F, involving the size of the opposing bunch and its intensity expressed through the nominal beam-beam parameter ξ_0. From M we obtain new values of μ and β-function, modified by collision. Let us consider equal intensities of the colliding bunches (equal ξ_0). After simple calculations [4], using a convenient notation: $x = 2\pi\xi_0$, $c = \cot 2\pi\nu$, $b_i = \beta^*/\beta_i$, and e_i for the normalized emittances (here $i = 1, 2$ refer to the two bunches in collision), we get equations on self-consistent β-functions in the special case of round beams:

$$
\begin{aligned}
b_1^2 &= 1 + 2xc\frac{b_2}{e_2} - x^2\frac{b_2^2}{e_2^2} \\
b_2^2 &= 1 + 2xc\frac{b_1}{e_1} - x^2\frac{b_1^2}{e_1^2}.
\end{aligned} \tag{1}
$$

The problem is periodic in ν with the period $1/2$, therefore we only consider $0 < \nu < 1/2$ in what follows.

We start with the case of constant emittances. Unequal solutions of (1) b_i correspond to the flip-flop situation. They are real and positive when $\nu \in (0, 1/4)$ and x belongs to the interval:

$$
\sqrt{\frac{2 + 3c^2 + c\sqrt{8 + 9c^2}}{2(1 + c^2)}} < x < c + \sqrt{1 + c^2}. \tag{2}
$$

For $\nu \in (1/4, 1/2)$ the inequalities should be reversed. Small ν are of predominant interest for a high beam-beam performance. One can obtain the threshold value of x for small ν by taking the limit $c \to \infty$ in the LHS of (2): $x_{th} = \sqrt{3}$.

This is a very large and unrealistic value of x, which corresponds to $\xi_0 \simeq 0.26$.

Another way to get x_{th} is the graphical method [6], applied to (1): we consider b_1 as function of b_2, and evaluate the derivative $\partial b_1/\partial b_2$ at the point of equal b_i, thus inspecting a possibility for unequal solutions to appear. Then the flip-flop threshold $x = x_{th}$ satisfies the equation:

$$
\left.\frac{\partial b_1}{\partial b_2}\right|_{b1=b2} = -1, \tag{3}
$$

yielding the same x_{th} as in the LHS of (2).

In contrast to the above solution, one may expect a *non-round* flip-flop state, e.g. a cross-shaped one: $b_{1x} = b_{2y}$, $b_{1y} = b_{2x}$. Using (3) we obtain this threshold:

$$
x \geq x_{th} = \sqrt{\frac{4 + 5c^2 + c\sqrt{24 + 25c^2}}{2(1 + c^2)}}
$$

It appears to be even higher than that for the round flip-flop state: the round beam shape seems to be "flip-proof", cf. [7].

3 RADIATION EMITTANCE

Assuming the emittances unchanged by collision, we see from the above section that the flip-flop thresholds are rather high in terms of ξ_0.

Let us now evaluate the radiation emittance of the bunch with a thin lens insertion at the IP, representing the linear effect of collision. Zero dispersion at the IP is assumed for simplicity.

The equilibrium emittance is determined by the one-turn average of the Courant-Snyder quadratic form with the dispersion function:

$$H(s) = \beta(s)\,\eta'(s)^2 + 2\alpha(s)\,\eta(s)\eta'(s) + \gamma(s)\,\eta(s)^2 .$$

The lens insertion modifies the Twiss parameters involved in $H(s)$ thus changing the radiation emittance. In the Floquet parametrization $H(s) = |W(s)|^2$, the appropriate Wronskian reads:

$$W(s) = \begin{vmatrix} \eta(s) & w(s) \\ \eta'(s) & w'(s) + i/w(s) \end{vmatrix} e^{i\phi},$$

and the modified Floquet function $w(s)\,e^{i\phi}$ should be decomposed via the basis of the unperturbed Floquet vectors at the IP, then propagated through the unperturbed arc to get the modified vector on the current azimuth of integration s. Thus we obtain the effect of collision:

$$\frac{H(s)}{H_0(s)} = \frac{1 + p\cot 2\pi\nu + p\csc 2\pi\nu \cos 2(\arg W_0(s) - \pi\nu)}{\sqrt{1 + 2\,p\cot 2\pi\nu - p^2}}$$

where $p = P\beta^*/2$ is the normalized strength of the lens, and the 0 subscript marks the quantities relevant to the unperturbed lattice.

The $1 + p\cot 2\pi\nu$ term in the numerator gives positive definite contribution to the radiation emittance, and compares to the result of [5]. But the 2nd term, proportional to $\cos 2(\arg W_0(s) - \pi\nu)$, depends on the arc lattice, and generally its contribution to the radiation integral does not vanish. It may well override the effect the 1st term in some particular lattices, resulting in a linear slope of either sign in the emittance dependence on ξ_0, contradicting to [2, 5].

4 MODELS OF EMITTANCE VARIATION

We can implement the above conclusion in simple models of variable emittance, to be used jointly with (1) for the self-consistent analysis.

The 1st model assumes the linear variation of emittance with the strength of the lens of the opposite bunch: we have then for the normalized emittances:

$$\begin{aligned} e_1 &= 1 + k\frac{b_2}{e_2} \\ e_2 &= 1 + k\frac{b_1}{e_1}, \end{aligned} \qquad (4)$$

k is the linear slope coefficient; it should be kept not too large for our model to be valid. We solve (4) for e_1 and e_2 first, substitute these solutions into (1) to obtain two equations on the two variables (b_1, b_2):

$$\begin{aligned} b_1^2 &= 1 + \frac{4b_2x(c(1 + (b_1 - b_2)k + \sqrt{D}) - b_2x)}{(1 + (b_1 - b_2)k + \sqrt{D})^2} \\ b_2^2 &= 1 + \frac{4b_1x(c(1 + (b_2 - b_1)k + \sqrt{D}) - b_1x)}{(1 + (b_2 - b_1)k + \sqrt{D})^2} \end{aligned} \qquad (5)$$

where $D = 4b_2k + (1 + (b_1 - b_2)k)^2$.

Now we may solve the problem using method [6]. With $k > 0$, the flip-flop situation appears only at high values of $x = 2\pi\xi_0$ (Fig.1). In this case the values of the self-consistent β-functions are small enough and the emittances exceed their nominal values.

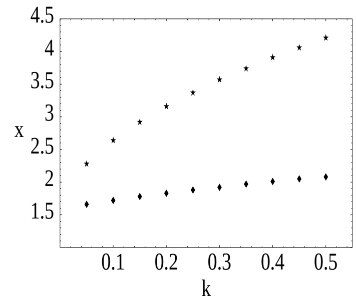


Figure 1: The flip-flop threshold x_{th} vs. the positive slope k in (5), $\nu = 0.01$ (top), $\nu = 0.1$ (bottom).

The case $k < 0$ is more interesting. There is some limiting value of x, which depends on the values of k and ν. If x is above this limit, the system (5) has no solutions. And when x is close to its threshold, there is a range of x, where (5) has two different solutions with equal b_i. We may avoid this situation by increasing the value of ν. Before x approaches its maximal value, (5) has one solution with equal b_i.

The 2nd model: we assume linear variation of the beam sizes with the strength of the lens of opposite bunch. After some calculations, this model is expressed by equations

$$\begin{aligned} e_1 &= b_1(1 + kx\frac{b_2}{e_2})^2, \\ e_2 &= b_2(1 + kx\frac{b_1}{e_1})^2, \end{aligned} \qquad (6)$$

reducible to 2 variables e_1/b_1 and e_2/b_2 only. Hence, we solve (6) for these and substitute into (1) to obtain solutions for b_i.

The resultant of two equations in (6) has simple factorization:

$$\begin{aligned} R &= (e_2^2 + 2b_2e_2kx + b_2^2k^2x^2 - b_2e_2k^2x^2) \times \\ &\quad (e_2^3 - b_2e_2^2 - 2b_2^2e_2kx - b_2^3k^2x^2). \end{aligned}$$

The first factor gives two solutions:

$$e_2 = \frac{1}{2}b_2kx(kx - 2 \pm \sqrt{kx(kx-4)}). \qquad (7)$$

So, for $k > 0$ the flip-flop threshold is high: we need $kx > 4$ for $e_{1,2}$ to be real. The second factor in R has one real root, if $k > 0$. It corresponds to normal solution $b_1 = b_2$.

Another situation is in the case of $k < 0$. Now (7) are always real and positive and correspond to the flip-flop solutions. After substitution (7) into (1) we get $b_{1,2}^2$ and require that they be positive; this yields the condition on existence of the flip-flop solutions:

$$x < x(k) = \tan \pi\nu\,(\cot \pi\nu + 1/k)^2\,; \quad \tan \pi\nu < -k\,.$$

These expressions indicate how to avoid the unwanted flip-flop situation: at some x and a given value of k in the linear dependence of emittances, we may raise the tune to shift it in the area of only equal solutions, crossing the flip-flop border shown in Fig 2.

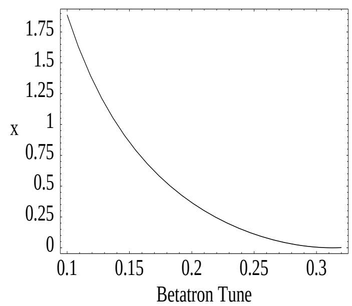
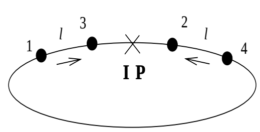


Figure 2: The flip-flop threshold x_{th} vs. the tune ν in the case of $k < 0$ in (6), $k = -0.8$. The flip-flop area lies under the curve.

5 EFFECT OF THE BUNCH LENGTH

In this section we present the constant emittance model, accounting for the effect of bunch length in collision by splitting either of the colliding bunches into 2 equal infinitely short ones spaced by l (in units of β_0):



The interaction process then has three phases: 1) collision of particles 2 and 3 at the IP; then 2a) collision of 2 and 1, and 2b) collision of 3 and 4 at the points positioned at the distance of $\mp l/2$ from the IP; finally, 3) collision of particles 1 and 4 at the IP. All values of the Twiss parameters α and β for each particle are taken at the IP and traced to the respective collision point.

From the matrix of one revolution for each particle M_i ($i = 1..4$) we get the new values of the phase advance μ, β- and α-functions and then obtain the equations on the self-consistent β-functions. This system is very complicated and can only be studied numerically. The first conclusion: if $l \neq 0$, there is no situation, when all β-functions are equal. We have the state in our system, when parameters of front and back particles are equal. We define the flip-flop situation when all the 4 parameters are different; the threshold for these solutions to appear is high (Fig.3). Therefore we conclude that the finite bunch length effect is not detrimental in the round beam case.

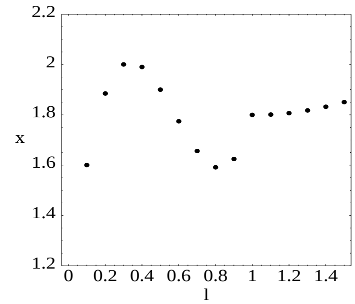


Figure 3: The flip-flop threshold x_{th} vs. l in the model of the bunch length.

6 CONCLUSION

The flip-flop effect is studied in terms of self-consistent β-function in the case, when the emittances of colliding bunches are influenced by the linear part of the beam-beam force. Evaluation of the radiation emittance of the bunch is presented in the case of one additional thin lens at the IP, with the emphasis on the term omitted in [5].

We have presented two models of variable emittances. One of them, when the emittance has a linear dependence on the strength of the lens of opposite bunch, gives high flip-flop thresholds in the area of positive slope k in (4) and no but equal sizes of colliding bunches if $k < 0$ and the beam intensity is below a certain limit. The second model (6) also predicts low flip-flop thresholds only when we assume $k < 0$, i.e. the size of the bunch is decreased by the force of opposite lens. We can avoid the flip-flop situation here by the working point manoeuvre. However, lattices with $k > 0$ seem to be generally preferable against the flip-flop effect.

The influence of the bunch length on the flip-flop effect thresholds in our simple model is weak.

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