Periodically-Focused Solutions to the Nonlinear Vlasov-Maxwell Equations for Intense Beam Propagation Through an Alternating-Gradient Quadrupole Field

Hong Qin and Ronald C. Davidson
Plasma Physics Laboratory
Princeton University, Princeton, NJ, 08543
Paul J. Channell
Los Alamos National Laboratory
Los Alamos, NM 87545

Abstract

This paper considers an intense nonneutral ion beam propagating in the z-direction through a periodic focusing quadrupole field with transverse focusing force, $\mathbf{F}_{foc} = -\kappa_q(s)(x\hat{\mathbf{e}}_x - y\hat{\mathbf{e}}_y)$, on the beam ions. A third-order Hamiltonian averaging technique using a canonical transformation is employed to transform away the rapidly oscillating terms. This leads to a Hamiltonian, $\mathcal{H}(\tilde{X},\tilde{Y},\tilde{X}',\tilde{Y}',s) = (1/2)(\tilde{X}'^2 + \tilde{Y}'^2) + (1/2)\kappa_{fq}(\tilde{X}^2+\tilde{Y}^2)+\psi(\tilde{X},\tilde{Y},s)$, in the transformed variables $(\tilde{X},\tilde{Y},\tilde{X}',\tilde{Y}')$, where the focusing coefficient κ_{fq} is constant, and many solutions and properties of the Vlasov-Maxwell system are well known.

1 INTRODUCTION

It is important to be able to investigate, based on the nonlinear Vlasov-Maxwell equations, the equilibrium and stability properties of general distribution functions for periodically-focused beams[1, 2, 3]. Despite its limited practical interest due to the unphysical distribution in phase space, the Kapchinskij-Vladimirskij (KV) beam equilibrium[1, 4, 5, 6], including its recent generalization to a rotating beam in a periodic focusing solenoidal field[7, 8], has been the *only* known periodically-focused equilibrium solution to the nonlinear Vlasov-Maxwell equations describing an intense beam propagating through a periodic focusing field. The difficulty of solving the nonlinear Vlasov-Maxwell system in general lies in the fact that the Hamiltonian for the motion of an individual beam particle is time dependent. Channell[9] and Davidson *et al*[10] have recently developed a third-order Hamiltonian averaging technique using a canonical transformation to average over the fast time scale associated with the betatron oscillations. This procedure is expected to be valid for sufficiently small phase advance ($\sigma \lesssim 60°$, say). In the present analysis, we apply this technique to the Vlasov-Maxwell system for intense beams propagating through a periodic focusing lattice. Under the thin-beam assumption, the applied transverse focusing force on a beam particle is $\mathbf{F}_{foc} = -\kappa_q(s)(x\hat{\mathbf{e}}_x - y\hat{\mathbf{e}}_y)$. The Vlasov-Maxwell equations for the distribution function $f_b(x,y,x',y',s)$ and the normalized self-field potential $\psi(x,y,s) = Z_b e\phi(x,y,s)/\gamma_b^3 m_b\beta_b^2 c^2$ can be expressed as[1, 7]

$$\begin{aligned}&\Big\{\frac{\partial}{\partial s} + x'\frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} - \Big(\kappa_q(s)x + \frac{\partial\psi}{\partial x}\Big)\frac{\partial}{\partial x'}\\ &-\Big(-\kappa_q(s)y + \frac{\partial\psi}{\partial y}\Big)\frac{\partial}{\partial y'}\Big\} f_b = 0\,,\end{aligned} \tag{1}$$

and

$$\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big)\psi = -\frac{2\pi K_b}{N_b}\int dx'dy' f_b\,. \tag{2}$$

Here,

$$K_b = \frac{2N_b Z_b^2 e^2}{\gamma_b^3 m_b \beta_b^2 c^2} \text{ and } N_b = \int dxdydx'dy' f_b \tag{3}$$

are the self-field perveance and the number of beam ions per unit axial length, respectively.

2 CANONICAL TRANSFORMATION

Because of the oscillatory time dependence of $\kappa_q(s)$, there is no general analytical method to solve the nonlinear Vlasov-Maxwell equations. However, we can average over the fast time scale associated with the betatron oscillations when the phase advance is sufficiently small. The averaging process is accomplished by introducing a canonical coordinate transformation from the laboratory coordinate system (x,y,x',y') to a new coordinate system (X,Y,X',Y'). In the laboratory coordinates, the single-particle Hamiltonian $H(x,y,x',y',s)$ is

$$H = \epsilon\left[\frac{1}{2}(x'^2+y'^2) + \frac{1}{2}\kappa_q(s)(x^2-y^2) + \psi(x,y,s)\right], \tag{4}$$

where ϵ is a small dimensionless parameter proportional to the focusing field strength. We use a near-identity canonical transformation $T : (x,y,x',y') \longmapsto (X,Y,X',Y')$ that is generated by a generating function of the Von Zeipel form, i.e.,

$$\begin{aligned}&S(x,y,X',Y',s) = xX' + yY'\\ &+\sum_{n=1}^{\infty}\epsilon^n S_n(x,y,X',Y',s)\,.\end{aligned} \tag{5}$$

Consequently, the transformed Hamiltonian in the new variables $\mathcal{H}(\tilde{X}, \tilde{Y}, \tilde{X}', \tilde{Y}', s)$ is given by

$$\mathcal{H} = \sum_{n=1}^{\infty} \epsilon^n \mathcal{H}_n = H + \frac{\partial}{\partial s} S(x, y, X', Y', s) \ . \qquad (6)$$

The corresponding coordinate transformation is given by

$$\begin{aligned} X &= \frac{\partial S}{\partial X'} = x + \sum_{n=1}^{\infty} \epsilon^n \frac{\partial}{\partial X'} S_n(x, y, X', Y', s) \ , \\ x' &= \frac{\partial S}{\partial x} = X' + \sum_{n=1}^{\infty} \epsilon^n \frac{\partial}{\partial x} S_n(x, y, X', Y', s). \end{aligned} \qquad (7)$$

The equations for Y and y' are similar in form.. We choose, order by order, the generating function S_n in such a way that $\mathcal{H}_n$ is independent of the fast time scale associated with oscillations in $\kappa_q(s)$, and solve for the coordinate transformation iteratively when S_n is known. Following the detailed algebra presented in Ref. [10], we obtain the transformed Hamiltonian correct to order ϵ^3,

$$\mathcal{H} = \frac{1}{2}(\tilde{X}'^2 + \tilde{Y}'^2) + \frac{1}{2}\kappa_{fq}(\tilde{X}^2 + \tilde{Y}^2) + \psi(\tilde{X}, \tilde{Y}, s) \ , \qquad (8)$$

where we have set $\epsilon = 1$. Here, κ_{fq} is defined in Eq. (11), and we have introduced the additional (canonical) fiber transformation to shifted velocity coordinates defined by

$$\begin{aligned} \tilde{X} = X \ , \ \ \tilde{X}' = X' - \langle \alpha_q \rangle X \ , \\ \tilde{Y} = Y \ , \ \ \tilde{Y}' = Y' + \langle \alpha_q \rangle Y \ . \end{aligned} \qquad (9)$$

Similarly, correct to order ϵ^3, we calculate the inverse coordinate transformation, $x = X + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3$, $x' = X' + \epsilon x_1' + \epsilon^2 x_2' + \epsilon^3 x_3'$, etc. Setting $\epsilon = 1$, this gives[10]

$$\begin{aligned} x(\tilde{X}, \tilde{Y}, \tilde{X}', \tilde{Y}', s) &= [1 - \beta_q(s)]\tilde{X} + 2\Big(\int_0^s ds \beta_q(s)\Big)\tilde{X}' \ , \\ x'(\tilde{X}, \tilde{Y}, \tilde{X}', \tilde{Y}', s) &= [1 + \beta_q(s)]\tilde{X}' + \Big\{ -\alpha_q(s) + \langle \alpha_q \rangle \\ &+ \langle \alpha_q \rangle \beta_q(s) - \alpha_q(s)\beta_q(s) - \Big(\int_0^s ds[\delta_q(s) - \langle \delta_q \rangle]\Big)\Big\}\tilde{X} \\ &+ \Big(\int_0^s ds \beta_q(s)\Big)\frac{\partial}{\partial \tilde{X}}\Big(\tilde{X}\frac{\partial \psi(\tilde{X}, \tilde{Y})}{\partial \tilde{X}} - \tilde{Y}\frac{\partial \psi(\tilde{X}, \tilde{Y})}{\partial \tilde{Y}}\Big) \ . \end{aligned} \qquad (10)$$

The coordinate transformation can be easily obtained by solving Eq. (10) for $\tilde{X}$ and $\tilde{X}'$ in terms of x and x'. The expressions for y and y' are identical in form to Eq. (10) provided we make the replacements $(x, x') \rightarrow (y, y')$ and $(\tilde{X}, \tilde{Y}, \tilde{X}', \tilde{Y}') \rightarrow (\tilde{Y}, \tilde{X}, \tilde{Y}', \tilde{X}')$ and reverse the signs of $\alpha_q(s)$ and $\beta_q(s)$. In the above equations, $\alpha_q(s)$, $\beta_q(s)$, and $\delta_q(s)$ are defined in terms of the lattice function $\kappa_q(s)$, which is assumed to have zero average, $\int_0^S ds \kappa_q(s) = 0$, and odd half-period symmetry with $\kappa_q(s - S/2) = -\kappa_q[-(s - S/2)]$. The definitions are given by

$$\begin{aligned} &\alpha_q(s) = \int_0^s ds \kappa_q(s) \ , \ \beta_q(s) = \frac{1}{S}\int_0^s ds[\alpha_q(s) - \langle \alpha_q \rangle] \ , \\ &\langle ... \rangle \equiv \frac{1}{S}\int_0^S ds(...) \ , \ \delta_q(s) = \alpha_q^2(s) - 2\kappa_q(s)\beta_q(s) \ , \\ &\kappa_{fq} = \langle \delta_q \rangle - \langle \alpha_q \rangle^2 = \frac{3}{S}\int_0^S ds[\alpha_q^2(s) - \langle \alpha_q \rangle^2] \ . \end{aligned} \qquad (11)$$

In addition, $\alpha_q(s)$ and $\langle \alpha_q \rangle$ are of order ϵ; $\beta_q(s)$ is of order ϵ^2; and $\langle \alpha_q \rangle \beta_q(s)$, $\alpha_q(s)\beta_q(s)$, $\big(\int_0^s ds \beta_q(s)\big)$, and $\big(\int_0^s ds[\delta_q(s) - \langle \delta_q \rangle]\big)$ are of order ϵ^3.

3 VLASOV-MAXWELL EQUATIONS IN THE TRANSFORMED VARIABLES

Because the transformation leading to the new Hamiltonian in Eq. (8) is canonical, the nonlinear Vlasov-Maxwell equations for the distribution function $F_b(\tilde{X}, \tilde{Y}, \tilde{X}', \tilde{Y}', s)$ and self-field potential $\psi(\tilde{X}, \tilde{Y}, s)$ in the transformed variables are given by

$$\begin{aligned} \Big\{ \frac{\partial}{\partial s} + \tilde{X}'\frac{\partial}{\partial \tilde{X}} + \tilde{Y}'\frac{\partial}{\partial \tilde{Y}} - \Big(\kappa_{fq}\tilde{X} + \frac{\partial \psi}{\partial \tilde{X}}\Big)\frac{\partial}{\partial \tilde{X}'} \\ - \Big(\kappa_{fq}\tilde{Y} + \frac{\partial \psi}{\partial \tilde{Y}}\Big)\frac{\partial}{\partial \tilde{Y}'}\Big\} F_b = 0 \ , \end{aligned} \qquad (12)$$

and

$$\left(\frac{\partial^2}{\partial \tilde{X}^2} + \frac{\partial^2}{\partial \tilde{Y}^2}\right)\psi = -\frac{2\pi K_b}{N_b}\int d\tilde{X}' d\tilde{Y}' F_b \ , \qquad (13)$$

where $\kappa_{fq} = const.$ is defined in Eq. (11). Variables in laboratory-frame coordinates can be obtained through the *pull-back* transformation $\tilde{T}^*$ associated with the coordinate transformation

$$\tilde{T} : (x, y, x', y') \longmapsto (\tilde{X}, \tilde{Y}, \tilde{X}', \tilde{Y}') \ . \qquad (14)$$

Here, $\tilde{T}^*$ pulls (transforms) functions on $(\tilde{X}, \tilde{Y}, \tilde{X}', \tilde{Y}')$ back into functions on (x, y, x', y'). For example, the distribution function transforms according to

$$\begin{aligned} &\tilde{T}^* : F_b(\tilde{X}, \tilde{Y}, \tilde{X}', \tilde{Y}', s) \longmapsto f_b(x, y, x', y', s) \\ &\equiv F_b(\tilde{T}(x, y, x', y'), s) \ . \end{aligned} \qquad (15)$$

In addition, we obtain the following pull-back equation for the beam density correct to order ϵ^3,

$$\begin{aligned} n_b(x, y, s) &= \int d\bar{x} d\bar{y} dx' dy' \ f_b \delta(\bar{x} - x)\delta(\bar{y} - y) \\ &= \int d\tilde{X} d\tilde{Y} d\tilde{X}' d\tilde{Y}' F_b \delta(\tilde{T}^{-1}\tilde{X} - x)\delta(\tilde{T}^{-1}\tilde{Y} - y) \\ &= \Big\{ \int d\tilde{X}' d\tilde{Y}' \ [1 - (x_2 + x_3)\frac{\partial}{\partial \tilde{X}} \\ &- (y_2 + y_3)\frac{\partial}{\partial \tilde{Y}}]F \Big\}_{(\tilde{X}, \tilde{Y}) \rightarrow (x, y)} \ . \end{aligned} \qquad (16)$$

Here, x_2, y_2 and x_3, y_3, defined by Eq. (10), are the second-order and third-order inverse coordinate transformations expressed as functions of $(\tilde{X}, \tilde{Y}, \tilde{X}', \tilde{Y}')$.

Because of the simple form of the Vlasov-Maxwell equations in the transformed variables, with constant focusing coefficient $\kappa_{fq} = const.$, a wide range of literature developed for the constant focusing case[1, 11, 12, 13] can be applied virtually intact in the transformed variables. For example, it is readily shown that any distribution function of the form

$$F_b^0(\tilde{X}, \tilde{Y}, \tilde{X}', \tilde{Y}') = F_b^0(\mathcal{H}^0) , \qquad (17)$$

where $\mathcal{H}^0 = (1/2)(\tilde{X}'^2 + \tilde{Y}'^2) + (1/2)\kappa_{fq}(\tilde{X}^2 + \tilde{Y}^2) + \psi^0(\tilde{X}, \tilde{Y})$ is the single-particle Hamiltonian, is an exact equilibrium solution to the Vlasov-Maxwell equations (12) and (13) with $\partial/\partial s = 0$. There is clearly enormous latitude[1, 7] in specifying the functional form of $F_b^0(\mathcal{H}^0)$ in the transformed variables, with equilibrium examples[10] ranging from the KV distribution, to the waterbag equilibrium, to thermal equilibrium, to mention a few examples. Once the functional form of $F_b^0(\mathcal{H}^0)$ is specified, and ψ^0 is calculated self-consistently from Eq. (13), periodically-focused equilibrium properties in the laboratory coordinates, such as the density profile and the transverse temperature profile, can then be determined by the pull-back transformation. For example, to the leading order, the density profile is of the form[10]

$$n_b(x, y, s) = n_b^0\left(\frac{x}{1 - \beta_q(s)}, \frac{y}{1 + \beta_q(s)}\right) , \qquad (18)$$

where $n_b^0(\tilde{X}, \tilde{Y}) = \int d\tilde{X}' d\tilde{Y}' \; F_b^0(\tilde{X}, \tilde{Y}, \tilde{X}', \tilde{Y}')$.

4 CONCLUSIONS

To summarize, the formalism developed here represents a powerful framework for investigating the kinetic equilibrium and stability properties of an intense nonneutral ion beam propagating through an alternating-gradient quadrupole field. First, the analysis applies to a broad class of equilibrium distributions $F_b^0(\mathcal{H}^0)$ in the transformed variables. Second, the determination of (periodically-focused) beam properties in the laboratory frame is relatively straightforward. Third, the analysis applies to beams with arbitrary space-charge intensity, consistent only with requirement for radial confinement of the beam particles by the applied focusing field $(\kappa_{fq}\beta_b^2 c^2 > \hat{\omega}_{pb}^2/2\gamma_b^2)$. Finally, the formalism can be extended[10] in a straightforward manner to the case of a periodic-focusing solenoidal field $\mathbf{B}_{sol}(\mathbf{x}) = B_z(s)\hat{\mathbf{e}}_z - (1/2)B_z'(s)(x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y)$, and to the case where weak nonlinear corrections to the focusing force are retained in the analysis.

5 ACKNOWLEGEMENT

This research was supported by the Department of Energy.

6 REFERENCES

[1] R. C. Davidson, *Physics of Nonneutral Plasmas* (Addison-Wesley Publishing Co., Reading, MA, 1990), and references therein.

[2] T. P. Wangler, *Principles of RF Linear Accelerators* (John Wiley & Sons, Inc., New York, 1998).

[3] M. Reiser, *Theory and Design of Charged Particle Beams* (John Wiley & Sons, Inc., New York, 1994).

[4] I. Kapchinskij and V. Vladimirskij, in *Proceedings of the International Conference on High Energy Accelerators and Instrumentation* (CERN Scientific Information Service, Geneva, 1959), p. 274.

[5] R. L. Gluckstern, in *Proceedings of the 1970 Proton Linear Accelerator Conference*, Batavia, IL, edited by M. R. Tracy (National Accelerator Laboratory, Batavia, IL, 1971), p. 811.

[6] T. -S. Wang and L. Smith, Particle Accelerators **12**, 247 (1982).

[7] R. C. Davidson and C. Chen, Particle Accelerators **59**, 175 (1998).

[8] C. Chen, R. Pakter, and R. C. Davidson, Phys. Rev. Lett. **79**, 225 (1997).

[9] P. J. Channell, Physics of Plasmas **6**, 982 (1999).

[10] R. C. Davidson, H. Qin, and P. J. Channell, to be published (1999).

[11] R. C. Davidson, W. W. Lee, and P. H. Stoltz, Phys. Plasmas **5**, 279 (1998).

[12] R. C. Davidson, Physical Review Letters **81**, 991 (1998).

[13] R. C. Davidson, Physics of Plasmas **5**, 3459 (1998).