Single Bunch Monopole Instability*

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Abstract

We study single bunch stability with respect to monopole longitudinal oscillations in electron storage rings. Our analysis is different from the standard approach based on the linearized Vlasov equation. Rather, we reduce the full nonlinear Fokker-Planck equation to a Schrödinger-like equation which is subsequently analyzed by perturbation theory. We show that the Haissinski solution [3] may become unstable with respect to monopole oscillations and derive a stability criterion in terms of the ring impedance.

1 INTRODUCTION

Single bunch longitudinal instability often limits the performance of electron storage rings. Theoretical analysis of this instability is usually based on the Fokker-Planck equation that includes the effects of both Hamiltonian and stochastic forces. The Hamiltonian part describes the synchrotron motion while radiation terms account for the much slower effects of the synchrotron radiation and define the beam size at low intensity. A stationary solution of the Fokker-Planck equation was first obtained in 1973 by J. Haissinski [3]. Since then much of the instability analysis was done utilizing the linearized Vlasov equation technique, where the Fokker-Plank equation is linearized with respect to the Haissinski solution. In this approach the Haissinski solution is also used to introduce the action-angle variables that make the Haissinski Hamiltonian independent of angle. The linearized Vlasov technique leads to the concept of azimuthal phase space modes, which are the components of the perturbation to the Haissinski solution with certain azimuthal symmetry. The first three of such modes are sketched in Fig. 1. Neglecting the possibility of several potential well minima we assume that action-angle variables can be defined uniformly across the whole plane.

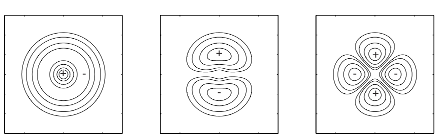

Figure 1: Example contour plots of the monopole, dipole and quadrupole modes

As seen from Fig. 1 the monopole mode is special because its physical space projection does not change significantly during a synchrotron period. This argues that radiation rather than Hamiltonian forces define the dynamics of this mode. Since the monopole mode has the same azimuthal structure as the Haissinski solution it is normally omitted from the standard linearized Vlasov analysis. There the radiation terms in the Fokker-Plank equation define the Haissinski solution and then only Hamiltonian terms remain in the linearized Vlasov equation. The possibility that a perturbation is monopole, but with radial structure different from the Haissinski solution, is neglected.

In this paper we are exploring the possibility that an instability can be associated with the monopole mode. Rather than extending the linearized Vlasov technique we find it more convenient to transform the Fokker-Plank equation to a Schrödinger-like equation and analyze the latter using the Haissinski solution as a basis. Advantages of this approach are that it is tractable and it allows us to use some well known facts about Schrödinger equation solutions.

We assume below that the monopole mode can be considered separately from other modes. The validity and consequences of this assumption are discussed in [1].

2 NOTATION AND BASIC EQUATIONS

For a relativistic bunch longitudinal dynamics is often described in dimensionless variables

$$x = z/\sigma_0, \quad p = -\delta/\delta_0, \quad \tilde{\tau} = \omega_{s0} t, \tag{1}$$

where z is the position of a particle with respect to the bunch centroid ($z > 0$ in the head of a bunch), δ is the relative energy spread, ω_{s0} is the synchrotron frequency, and the subscript "0" refers to zero-current quantities. The Fokker-Planck equation for the distribution function $\rho(x, p, \tilde{\tau})$ can be written [2] as

$$\frac{\partial \rho}{\partial \tilde{\tau}} + \{H, \rho\}_{p,x} = \frac{\gamma_d}{\omega_{s0}} \frac{\partial}{\partial p} \left(\frac{\partial \rho}{\partial p} + p\rho \right), \tag{2}$$

where $\{...\}$ denotes the Poisson brackets, γ_d is the radiation damping rate, $H(x, p, \tilde{\tau})$ is the self-consistent Hamiltonian

$$H(x, p, \tilde{\tau}) \equiv \frac{p^2}{2} + \frac{x^2}{2} + \Lambda \int dx' dp' \rho(x', p', \tilde{\tau}) S(x' - x), \tag{3}$$

and ρ is normalized to 1. We have neglected the nonlinearities of RF potential and defined the parameter

$$\Lambda \equiv N r_0 (C \gamma \alpha \delta_0^2)^{-1}, \tag{4}$$

where N is the number of particles in a bunch, r_0 is the classical electron radius, C is the ring circumference and α is the momentum compaction. We have also defined a dimensionless function $S(x) \equiv \sigma_0 \int_0^x dx' W(\sigma_0 x')$ in terms of the wakefield $W(z)$ for two particles separated by z.

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The Fokker-Planck equation (2) has a steady-state Haissinski solution [3]

$$\rho_H(x,p) = Z_H e^{-H_H(x,p)}. \tag{5}$$

where Z_H is a normalizing factor and H_H is defined by (3) with ρ replaced by ρ_H.

Canonical transformation from x, p to action-angle variables J, ϕ can be defined to make the Hamiltonian H_H phase independent, $H_H(x,p) \to H_H(J)$. Ignoring nonzero azimuthal modes by assuming $H = H(J,\tilde{\tau})$, $\rho = \rho_0(J,\tilde{\tau})$ the dynamics of the monopole mode is described by (2) transformed to J, ϕ variables and averaged over phase. This can be done using the invariance of the Poisson brackets [4]. Introducing the diffusion coefficient

$$D(J) \equiv J/\omega_H(J), \tag{6}$$

and renormalizing time to the damping rate $\tau \equiv \gamma_d t$ we can transform ([1]) equation (2) to the form

$$\frac{\partial \rho}{\partial \tau} = \frac{\partial}{\partial J}\left(D(J)\left[\frac{\partial \rho}{\partial J} + \omega(J,\tau)\rho\right]\right), \tag{7}$$

where $\omega_H(J) \equiv \frac{\partial H_H(J)}{\partial J}$ and $\omega(J,\tau) \equiv \frac{\partial H(J,\tau)}{\partial J}$.

3 TRANSFORMATION TO A SCHRÖDINGER-LIKE EQUATION

The Fokker-Planck equation (7) has a standard form that permits transformation to a Schrödinger-like equation [5]. Let us introduce a new independent variable

$$y \equiv y(J) = \int_0^J dJ'/\sqrt{D(J')} \tag{8}$$

and two functions

$$f(y,\tau) \equiv \frac{1}{\sqrt{D(J(y))}} e^{\Phi(y,\tau)/2} \rho(J(y),\tau), \tag{9}$$

$$\Phi(y,\tau) \equiv H(J(y),\tau) - (1/2)\ln D(J(y)), \tag{10}$$

where $J(y)$ is given implicitly by (8). Now the Fokker-Planck equation (7) takes the form

$$\frac{\partial f}{\partial \tau} = \frac{\partial^2 f}{\partial y^2} - U_S(y,\tau) f + \frac{1}{2}\dot{\Phi}(y,\tau) f, \tag{11}$$

where

$$U_S(y,\tau) \equiv [\Phi'(y,\tau)/2]^2 - \Phi''(y,\tau)/2 \tag{12}$$

and dot and prime denote partial derivatives with respect to τ and y. Eq. (11) is nonlinear since Φ is related to f by a self-consistency condition ([1]) that follows from (3).

Note, that $f_H(y) \equiv Z_H \sqrt[4]{D(J(y))} e^{-H_H(J(y))/2}$ is the steady-state solution of (11) and it corresponds to the Haissinski solution. Without the last term (11) can be thought of as a Schrödinger equation for a particle in the potential well $U_S(y,\tau)$. Since this term is zero for the Haissinski solution, one can neglect it for solutions that are close to f_H. This includes the case of the early time behavior of a system initialized with the Haissinski distribution at $\tau = 0$.

4 SCHRÖDINGER EQUATION ANALYSIS

After neglecting the $\dot{\Phi}$ term equation (11) reads

$$\frac{\partial f}{\partial \tau} = \frac{\partial^2 f}{\partial y^2} - U(y,\tau) f. \tag{13}$$

First, we solve a linear problem for which $\omega(J) = \omega^0$ is a constant. In this case $y = 2\sqrt{\omega^0 J}$ and the Schrödinger potential is simply

$$U_S^0(y) = \frac{y^2}{16} - \frac{1}{2} - \frac{1}{4y^2}, \tag{14}$$

which makes (13) a solvable eigenvalue problem. The solution is

$$f^0(y,\tau) = \sum_{m=0}^{\infty} \psi_m^0(y) e^{-\lambda_m^0 \tau}, \tag{15}$$

$$\psi_m^0(y) = (y/2)^{1/2} e^{-y^2/8} L_m(y^2/4), \tag{16}$$

where $\lambda_m^0 = m = 0, 1, 2, ...$ are the eigenvalues and L_m is the Laguerre polynomial of order m. As expected, the linear problem does not have unstable solutions. Any initial distribution exponentially approaches the Haissinski solution $\psi_0^0(y)$ on the time-scale defined by radiation damping.

For the general case, $\omega(J) \neq const$, asymptotic behavior of the solutions of (11) is described by the solutions to the linear problem (14),(16). In spite of the singularity in the potential the eigenvalues λ_m are bounded from below [1]. Furthermore, because $f_H(y)$ does not have zeros, this solution has the lowest eigenvalue $\lambda_0 = 0$ and the rest of λ_m are positive. Therefore, in this approximation, the Haissinski solution is stable.

5 PERTURBATION THEORY

How much the conclusion above depends on the assumption that the $\dot{\Phi}$ term in (11) is negligible can be analyzed by a perturbation technique. The approach is summarized below and the details can be found in [1]. We assume small deviation from the Haissinski solution. This deviation is expanded over $\psi_m(y)$ that are orthogonal eigenfunctions of (13). Now (11) together with the self-consistency condition result in an infinite linear system for the expansion coefficients. Looking for exponentially varying solutions $\propto e^{\mu\tau}$ transforms this system to a matrix equation. Its solutions are given by the roots of the determinant for the infinite matrix

$$M \equiv \delta_{n,k} + 2\Lambda \frac{\kappa_{n,k}\lambda_k}{\mu + \lambda_k}, \tag{17}$$

where

$$\kappa_{n,k} = i\frac{2\sigma_0}{Z_0} \int \frac{d\nu}{2\pi} \frac{Z(\nu)}{\nu} F_n(\nu) F_k^*(\nu), \tag{18}$$

$$F_n(\nu) \equiv \int dJ d\phi \frac{\psi_n(y(J))}{\sqrt[4]{D(J)}} e^{-\frac{H_H(J)}{2} + i\nu\sigma_0 x(J,\phi)/c} \tag{19}$$

and Z_0 is the impedance of free space. Positive roots $\mu > 0$ mean instability to monopole excitation of a bunch.

Since the off-diagonal terms of $\kappa_{n,k}$ are small and the others quickly converge to zero, a good approximation for the roots μ can be found by truncating the matrix M. If we truncate it to the lowest nontrivial rank 2 then zero determinant occurs for $\mu = -\lambda_1(1 + 2\Lambda\kappa_{1,1})$. Because $\lambda_1 > 0$, this root is positive when

$$2\Lambda\kappa_{1,1} < -1, \qquad (20)$$

and this may be viewed as the criterion for the onset of the monopole instability.

The sign of $\kappa_{1,1}$ is given by the odd part of the impedance, $ImZ(\nu)$ which is negative for inductive impedance. As a result, for the most common case of positive α and inductive impedance, $\Lambda\kappa_{1,1} > 0$ and the Haissinski solution is stable.

The situation is different for negative momentum compaction or in the case of capacitive impedance. Each of these have been proposed to get shorter bunches and as a remedy against longitudinal instabilities. For illustration, we use a broadband ($Q = 1$) resonator impedance model (e.g. [6]) with shunt impedance R_s and resonator frequency ω_R. Using (18)-(19) we numerically compute the quantity $2\Lambda\kappa_{1,1}$ as a function of normalized bunch length $\sigma \equiv \omega_R\sigma_0/c$ at intensity $I \equiv 4\pi\Lambda R_s/Z_0 = -1$, where minus is due to $\alpha < 0$. The result and the threshold given by (20) are plotted in Fig. 2. It shows that a bunch is monopole unstable at this intensity provided σ_0 exceeds about 1/12 of the resonator wavelength. Note, that this intensity is not too high. For example, for $\sigma = 3$, it only leads to about 5% increase in the incoherent frequency spread [1].

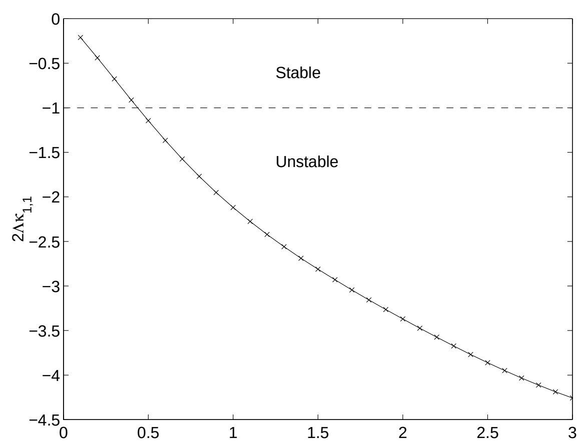

Figure 2: Monopole instability criterion (20) for broadband resonator impedance for $\alpha < 0$ and intensity $|I| = 1$.

6 DISCUSSION

We have investigated single bunch stability with respect to longitudinal monopole oscillations. These oscillations may become unstable as a result of an imbalance between radiation excitation and damping. Since this effect falls beyond the scope of the linearized Vlasov approach, we employed a different method that has not been used for instability analysis. This method involves the transformation of the phase-averaged Fokker-Planck equation to a Schrödinger-like equation which is analyzed by perturbation analysis.

Utilizing this technique we have obtained a criterion, (20) for the onset of monopole instability. We have found that this instability does not appear in the most common case of storage ring operation with positive momentum compaction when the impedance is largely inductive. However, for $\alpha < 0$ bunches may become monopole unstable at modest intensity. We expect a similar behavior for the case of predominantly capacitive impedance and $\alpha > 0$.

The monopole instability could be one of the factors preventing high current operation of storage rings with negative momentum compaction. Many attempts of such operation have been tried (e.g. [7], [8]) mainly to shorten a bunch and to avoid the microwave instability [6]. Usually only the static bunch shape and the energy spread measurements are reported and it is hard to infer what particular effect was the limitation. However, in some cases, it appears that there is something other than the microwave instability, because the threshold increase predicted [9] for this instability is not observed. An evidence of monopole instability might include growth of the longitudinal beam size, in the absence of synchrotron sidebands to the rotation harmonics of a beam position monitor signal.

We hope that the technique described in this paper can be applied to other problems in accelerator physics that lead to the one dimensional Fokker-Planck equation.

7 ACKNOWLEDGMENTS

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