WAKEFIELDS OF BUNCHED BEAMS IN RINGS WITH RESISTIVE WALLS OF FINITE THICKNESS

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Abstract

We present a thorough analysis and a full solution in the frequency domain, for the wake-fields of a (bunched)beam in a pipe with walls of finite conductivity and thickness, for the simplest pipe-geometry (circular). The wake-field multipoles for a multi-bunch beam in a circular ring are computed in analytic form, displaying the wake-field dependence on wall conductivity and thickness.

INTRODUCTION

Wake fields describe the interaction between a particle beam and the surrounding pipe wall. For perfectly conducting pipes and ultrarelativistic motion $(v = c)$ wakefields are negligibile. In the realistic case of walls of finite conductivity, and/or relatively low values of the relativistic factor γ, occurring, e.g., at injection, wake fields might be quite relevant. In addition, for low revolution frequencies, the finite thickness of the pipe wall should be properly taken into account [1]. Much has been written on the subject of wake fields, since the early work of Piwinski [2], who first studied the opposite limiting cases of a metal-coated ceramic vacuum chamber, where the coating is much thinner than the EM penetration depth, and of a homogeneous conducting pipe, much thicker than the EM penetration depth. Palumbo and Vaccaro extended Piwinsky's results for this latter case, by computing higher order wake-field multipoles [3]. Chao first gave a formula which fully exploits the dependence of the wakefield on the pipe wall thickness, but his analysis was restricted to the monopole term [4]. More recently, Ohmi and Zimmerman presented a thorough analysis of the subrelativistic effect [5]. Finally, Yokoya and Shobuda studied the finite-conductivity, finite-thickness pipe-wall problem, in the frame of a transmission line analogy, which can be applied to beam pipes with general transverse geometry and multi-layered walls, in the limit where the EM skin depth is much smaller than the (smallest) pipe transverse dimension [6]. In [7] we computed the fields of a (bunched) beam in a pipe with walls of finite conductivity and thickness, for the simplest pipe-geometry (circular). We solved the problem by computing the Fourier transform of the wake potential Green's function produced by a point particle running at constant velocity $\beta c\hat{u}_z$, at a distance r_o off axis of a circular cylindrical pipe with radius b, wall conductivity σ and thickness Δ.

The solution found is exact but complicated, so that in most cases of practical interest one has to resort to suitable limiting forms. In this paper we introduce a number of asymptotic approximations appropriate, in particular, to LHC (Large Hadron Collider) and DAFNE, whose relavant figures are collected in Table I.

THE GREEN'S FUNCTION

In [7] we obtained the Green's function for an off-axis point particle running at distance r_0 from the axis of a circular pipe of radius b with finite conductivity σ and thickness Δ, viz.:

$$\tilde{G}_m(k,r,r_0) = \tilde{G}_m^{\infty}(k,r,r_0) + \frac{q_o}{2\pi\epsilon_o}\frac{I_m(k'r_0)I_m(k'r)}{bk'I_m(k'b)}\frac{N(k)}{D(k)}, \tag{1}$$

where

$$\tilde{G}_m^{\infty}(k,r,r_0) = \frac{q_o}{2\pi\epsilon_o}\left\{A(k,r,r_0) - \frac{I_m(k'r_0)}{I_m(k'b)}K_m(k'b)I_m(k'r)\right\}. \tag{2}$$

In Eq. (2) $k' = k/\gamma$, $\tilde{G}_m^{\infty}$ is the solution of the wave equation corresponding to the perfectly conducting pipe, $A(\cdot)$, $N(k)$ and $D(k)$ are:

$$A(k,r,r_0) = \begin{pmatrix} K_m\left(\frac{kr}{\gamma}\right) I_m\left(\frac{kr_0}{\gamma}\right) & r_0 \le r \le b, \\ K_m\left(\frac{kr_0}{\gamma}\right) I_m\left(\frac{kr}{\gamma}\right) & r \le r_0, \end{pmatrix} \tag{3}$$

$$\begin{aligned} N(k) = {} & \bar{k}^2 K'_m(k'd)\left[I_m(\bar{k}b)K_m(\bar{k}d) - I_m(\bar{k}d)K_m(\bar{k}b)\right] + \\ & + \eta\,\bar{k}k'K_m(k'd)\left[K_m(\bar{k}b)I'_m(\bar{k}d) - I_m(\bar{k}b)K'_m(\bar{k}d)\right], \end{aligned} \tag{4}$$

$$\begin{aligned} D(k) = {} & \bar{k}^2 I'_m(k'b)K'_m(k'd)\left[I_m(\bar{k}b)K_m(\bar{k}d) - I_m(\bar{k}d)K_m(\bar{k}b)\right] \\ & + \eta\bar{k}k'I_m(k'b)K'_m(k'd)\left[K'_m(\bar{k}b)I_m(\bar{k}d) - I'_m(\bar{k}b)K_m(\bar{k}d)\right] \\ & + \eta\bar{k}k'K_m(k'd)I'_m(k'b)\left[I'_m(\bar{k}d)K_m(\bar{k}b) - K'_m(\bar{k}d)I_m(\bar{k}b)\right] \\ & + \eta^2 k'^2 I_m(k'b)K_m(k'd)\left[I'_m(\bar{k}b)K'_m(\bar{k}d) - I'_m(\bar{k}d)K'_m(\bar{k}b)\right] \end{aligned} \tag{5}$$

with $k' = k/\gamma$, $d = b + \Delta$ and

$$\eta = \frac{Z_o\sigma}{ik\beta} - 1. \tag{6}$$

It can be checked that Eq.(1) reduces to the solution obtained in [3] in the limit $d \to \infty$ of an infinitely thick wall.

ASYMPTOTIC APPROXIMATIONS

In most cases of practical interest, one may resort to suitable (asymptotic) limiting forms, since many problem-specific (dimensionless) parameters are either very large or very small.

Large Parameters

The following inequality always holds in view of the assumed beam spectral features:

$$\left|\bar{k}b\right|=\left|\sqrt{k'^2-i\sigma\beta kZ_o}\right|b\sim\left|\sqrt{-i\sigma\beta kZ_o}\,b\right|\equiv\left|\frac{b}{\delta_{wall}}\right|\gg 1, \tag{7}$$

where

$$\delta_{wall}=(-i\sigma\beta kZ_o)^{-1/2} \tag{8}$$

is the electromagnetic skin depth. One has also $|\bar{k}d| \gg 1$, since $d \gtrsim b$. Note also that, within the useful spectral ranges discussed above, one has from Eq. (6):

$$\eta \simeq -i\frac{Z_o\sigma}{k\beta}. \tag{9}$$

Accordingly, using the well known large-argument forms of the (modified) Bessel functions:

$$I_m(z)\sim\frac{e^z}{\sqrt{2\pi\, z}},\quad K_m(z)\sim\sqrt{\frac{\pi}{2z}}e^{-z}, \tag{10}$$

for $I_m(\cdot)$ and $K_m(\cdot)$ with arguments $\bar{k}b$ and $\bar{k}d$ in Eq. (1), one gets a simpler form for both $N(k)$ and $D(k)$, viz.:

$$N(k)=-\bar{k}^2K'_m(k'd)\sinh\bar{k}\Delta+\eta k'\bar{k}K_m(k'd)\cosh\bar{k}\Delta, \tag{11}$$

$$D(k)=\sinh\bar{k}\Delta\left[k'^2\eta^2I_m(k'b)K_m(k'd)-\bar{k}^2I'_m(k'b)K'_m(k'd)\right]$$
$$+\eta k'\bar{k}\cosh\bar{k}\Delta\left[I'_m(k'b)K_m(k'd)-I_m(k'b)K'_m(k'd)\right]. \tag{12}$$

Small Parameters

Let us next discuss the asymptotic limit:

$$|k|b/\gamma\sim|k|d/\gamma \ll 1. \tag{13}$$

For reasons which will be clarified soon, it is convenient to discuss separately the monopole $(m=0)$ and multipole $(m\geq 1)$ terms.

- The Monopole Term $(m=0)$. In the limit Eq. (13), one uses the zero-th order modified Bessel functions approximation valid for small arguments [8]:

$$I_0(\zeta)\sim 1,\quad K_0(\zeta)=-log(\zeta), \tag{14}$$

and hence the monopole term in Eq. (1) using Eq.s (11),(12) can be written

$$\tilde{G}_0(k,r,r_0)=\tilde{G}_0^{\infty}(k,r,r_0)+$$
$$\frac{q_o}{2\pi\epsilon_o}\frac{\gamma^2}{bk^2}\left[\frac{b}{2}+\eta\,\delta_{wall}\,\coth(\Delta/\delta_{wall})\right]^{-1}. \tag{15}$$

For a *very thick pipe wall*, $|\bar{k}\Delta|\sim|\Delta/\delta_{wall}|\gg 1$, whence $|\coth(\Delta/\delta_{wall})|\sim 1$, and Eq. (15) becomes:

$$\tilde{G}_0(k,r,r_0)=\tilde{G}_0^{\infty}(k,r,r_0)+\frac{q_o}{2\pi\epsilon_o b}\frac{\gamma^2}{k^2}\left(\frac{b}{2}+\eta\delta_{wall}\right)^{-1} \tag{16}$$

which, in the further limit (appropriate, e.g., both for LHC and DAFNE):

$$\left|2\eta\frac{\delta_{wall}}{b}\right|\gg 1, \tag{17}$$

yields the known result [3]:

$$\tilde{G}_0(k,r,r_0)=\tilde{G}_0^{\infty}(k,r,r_0)+\frac{q_o\beta\gamma^2}{2\pi\epsilon_o b}(1+i)\sqrt{\frac{\beta}{2\sigma Z_o k}}. \tag{18}$$

For a *finite-thickness pipe wall*, $|\,\Delta/\delta_{wall}\,|\geq 1$, in the same limit Eq. (17), Eq. (15) yields:

$$\tilde{G}_0(k,r,r_0)=\tilde{G}_0^{\infty}(k,r,r_0)+$$
$$\frac{q_o\beta\gamma^2}{2\pi\epsilon_o b}(1+i)\sqrt{\frac{\beta}{2\sigma Z_o k}}\tanh(\Delta/\delta_{wall}). \tag{19}$$

This latter, in the limit of infinite wall thickness, $|\Delta/\delta_{wall}|\to\infty$, gives back Eq. (18). The relative error produced by using Eq. (19) in place of Eq. (1) is shown in Fig.1 as a function of kb.

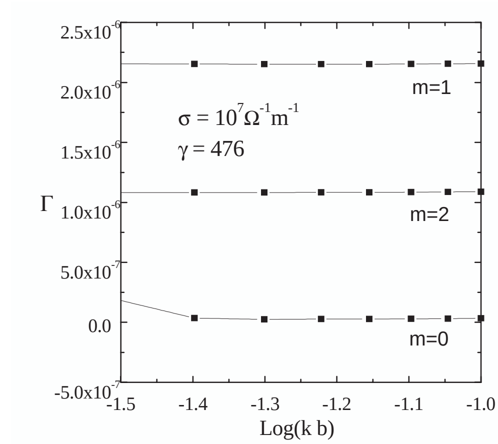

Figure 1: Relative error Γ on $(2\pi\epsilon_0/q)[\tilde{G}_m-\tilde{G}_m^{\infty}]$ versus kb after assuming $kb \ll 1$ and using Eq.s (19), (24) in place of Eq. (1); monopole, dipole and quadrupole terms (m=0,1,2).

- Multipole Terms **(**$m\geq 1$**).** In the asymptotic limit $|k|b/\gamma\ll 1$, $|k|d/\gamma\ll 1$ one uses in Eq.s (2), (1), (11) and (12) the small-argument asymptotic form of the modified Bessel functions of m-order [8]:

$$I_m(\zeta)\sim\left(\frac{\zeta}{2}\right)^m\frac{1}{m!},$$
$$K_m(\zeta)\sim\frac{(m-1)!}{2}\left(\frac{\zeta}{2}\right)^{-m},\quad(m>0). \tag{20}$$

Hence, from (2):

$$\tilde{G}_m^{\infty}(r,r_0)\approx\tilde{G}_m^{free\ space}(r,r_0)-\frac{q_o}{2\pi\epsilon_o}\frac{1}{2m}\left(\frac{rr_o}{b^2}\right)^m, \tag{21}$$

where

$$\tilde{G}_m^{free\ space}(r,r_0) \approx \frac{q_o}{2\pi\epsilon_o}\frac{1}{2m}\left(\frac{rr_o}{b^2}\right)^m R(r,r_o), \quad (22)$$

$$R(r,r_o) = \begin{cases} (r_0/r)^m & r_0 \le r \le b, \\ (r/r_0)^m \ , & r \le r_o, \end{cases} \quad (23)$$

and, from Eq.s (1),(11) and (12):

$$\tilde{G}_m(k,r,r_0) = \tilde{G}_m^{\infty}(k,r,r_0) + \frac{q_o}{2\pi\epsilon_o}\frac{1}{m}\left(\frac{rr_o}{b^2}\right)^m \cdot$$

$$\cdot\left[1+\frac{k^2\eta b}{m\bar{k}\gamma^2}\tanh(\bar{k}\Delta)\frac{\frac{k^2\eta d}{m\bar{k}\gamma^2}+\coth(\bar{k}\Delta)}{\frac{k^2\eta d}{m\bar{k}\gamma^2}+\tanh(\bar{k}\Delta)}\right]^{-1}. \quad (24)$$

which, using Eq.s (8), (9) can be equally written:

$$\tilde{G}_m(k,r,r_0) = \tilde{G}_m^{\infty}(k,r,r_0) + \frac{q_o}{2\pi\epsilon_o}\frac{1}{m}\left(\frac{rr_o}{b^2}\right)^m \cdot \quad (25)$$

$$\cdot\left[1+\frac{b/\delta_{wall}}{m\beta^2\gamma^2}\tanh\left(\frac{\Delta}{\delta_{wall}}\right)\frac{\frac{d/\delta_{wall}}{m\beta^2\gamma^2}+\coth\left(\frac{\Delta}{\delta_{wall}}\right)}{\frac{d/\delta_{wall}}{m\beta^2\gamma^2}+\tanh\left(\frac{\Delta}{\delta_{wall}}\right)}\right]^{-1}.$$

The relative error produced by using Eq. (24) in place of Eq. (1) for m=1,2 is shown in Figure 1.

As expected, the error increases with kb, but remains very small throughout the meaningful spectral range. Similar to the monopole term case, for a *very thick pipe wall* , one has $|\bar{k}\Delta| \sim |\Delta/\delta_{wall}| \gg 1$, and hence $\sinh\bar{k}\Delta \sim \cosh\bar{k}\Delta$. Thus Eq. (24) becomes:

$$\tilde{G}_0(k,r,r_0) = \tilde{G}_0^{\infty}(k,r,r_0) + \frac{q_o}{2\pi\epsilon_o m}\left(\frac{rr_o}{b^2}\right)^m\left(1+\frac{b/\delta_{wall}}{m\beta^2\gamma^2}\right)^{-1}. \quad (26)$$

The *finite-thickness pipe wall* , $|\Delta/\delta_{wall}| \ge 1$ case, will be now discussed with reference to a number of limiting cases relevant to our applications.

- **LHC** . In the Large Hadron Collider one has:

$$\left|\frac{b/\delta_{wall}}{\beta^2\gamma^2}\right| \ll 1, \quad \left|\frac{d/\delta_{wall}}{\beta^2\gamma^2}\right| \ll 1. \quad (27)$$

Accordingly, for not-too-small values of $|\Delta/\delta_{wall}|$,

$$\tilde{G}_m(k,r,r_o) = \tilde{G}_m^{\infty}(k,r,r_o) \ +$$

$$\frac{q_o}{2\pi\epsilon_o}\frac{1}{m}\left(\frac{rr_o}{b^2}\right)^m\left[1-\frac{b/\delta_{wall}}{m\beta^2\gamma^2}\coth\left(\frac{\Delta}{\delta_{wall}}\right)\right]. \quad (28)$$

Equation (28) reproduces the limit form of Eq. (26) under Eq. (27) provided $\Delta \gg |\delta_{wall}|$. In the extreme limiting case $|\Delta/\delta_{wall}| \ll 1$ the expression in square brackets in Eq. (24) becomes simply $(1+b/d)^{-1}$, so that using (21), one has:

$$\tilde{G}_m(r,r_0) \approx \tilde{G}_m^{free\ space}(r,r_0) -$$

$$\frac{q_o}{2\pi\epsilon_o}\frac{1}{2m}\left(\frac{rr_o}{b^2}\right)^m\left[1-2\left(1+\frac{b}{d}\right)^{-1}\right] \quad (29)$$

which reduces to the free-space term, if $\Delta \to 0$, i.e. $d \to b$, as expected.

- **Ultrashort Bunch Machines** . In ultrashort bunch machines, including, e.g., DAFNE, one has (Table I):

$$\left|\frac{b/\delta_{wall}}{m\beta^2\gamma^2}\right| \gg 1. \quad (30)$$

Accordingly, for not-too-small values of $|\Delta/\delta_{wall}|$,

$$\tilde{G}_m(k,r,r_0) = \tilde{G}_m^{\infty}(k,r,r_0) +$$

$$\frac{q_o}{2\pi\epsilon_o}\left(\frac{rr_o}{b^2}\right)^m \beta^2\gamma^2\frac{\delta_{wall}}{b}\coth(\Delta/\delta_{wall}). \quad (31)$$

In the extreme limiting case $|\Delta/\delta_{wall}| \ll 1$ the expression in square brackets in Eq. (25) becomes simply $(1+b/d)^{-1}$, so that using (21), one has:

$$\tilde{G}_m(r,r_0) \approx \tilde{G}_m^{free\ space}(r,r_0) -$$

$$\frac{q_o}{2\pi\epsilon_o}\frac{1}{2m}\left(\frac{rr_o}{b^2}\right)^m\left[1-2\left(1+\frac{b}{d}\right)^{-1}\right] \quad (32)$$

which reduces to the free-space term, if $\Delta \to 0$, i.e. $d \to b$, as expected.

Design parameters	LHC	DAFNE
Circumference L_c [m]	26658	97.69
# of bunches N_b	2835	120
Bunch length σ_s [cm]	$(7 \div 13)$	2
Lorentz factor γ	500÷7000	1000
Pipe diameter [cm]	3	10
Wall thickness [mm]	0.05 (Cu) , 1 (SS)	2 (Al)
Wall conductivity $[\Omega^{-1}m^{-1}]$	$(10^7 \div 10^{10})$	$3.4 \cdot 10^7$
Circulation frequency [MHz]	$11.2455 \cdot 10^{-3}$	368.26

Table I

CONCLUSIONS

In this paper the general exact Green's function for an (off-axis) multi-bunch beam in a circular pipe with finite wall conductivity and thickness was applied, using appropriate asymptotic limiting forms, to compute the wake field multipoles for the different paradigm cases of LHC and DAFNE. More or less trivial extensions include more complicated geometries (e.g., elliptical, square).

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