ACCELERATION AND FOCUSING AS OPTIMAL CONTROL FOR DYNAMIC SYSTEMS

Z. Parsa, Brookhaven National Laboratory, New York, USA
V. Zadorozhny*, Institute of Cybernetics, Kiev, Ukraine

Abstract

In this report we develop an approach to acceleration and focusing using some concepts from optimal control theory and L-moment problem. Several scenarios have been discussed earlier [see Z. Parsa and V. Zadorozhny, proceedings PAC -03, EPAC -02]. Those are given here in more detail in a simulation of acceleration and focusing for a longitudinal motion. Individual acceleration scenarios are considered in the framefork of the approach presented.

PROBLEM STATEMENT

A new approach to studying a nonlinear bunched beam dynamics based on the self-consistent Vlasov – Maxwell equations and arguments from optimal control theory is considered. An interesting property of Maxwell equations first stated by V.I. Zubov is employed. The property consists in the following. Given a specified beam motion in R^3 there exist electric and magnetic fields which realize this motion. This property along with some mathematical aspect of optimal control theory with a given quality criterion allows to construct in certain cases a solution to the problem of focusing and acceleration for charged particle beams. The approach can be regarded as a development of the known algorithm by R.C. Davidson.

This approach supposes the following main steps:

a) We have to find the Vlasov distribution function $f(t,x,v)$ so that the given transport conditions are satisfied. Those may include requirements on an acceleration, current density, value of focus, minimum of some criterion and so on. The idea is to look at this problem as to some optimal control problem [1]. The Vlasov equation is transformed into a Fredholm equation [2], [3] and then we can find $f(t,x,v)$ and dispersion waves.

b) Now we can construct a beam density $\rho(t,x)$ and a beam current density $j(t,x)$ according to Maxwell equations. Thus, there exist such fields E and H which provide a longitudinal motion of a beam according to the fixed law of motion.

Based on results of Zubov and Halmos this approach makes it feasible to apply the direct Lyapunov method to nonlinear problems for which an empirical method of constructing a Lyapunov function generates a certain kernel operator in a domain of its asymptotic stability, and then the Lyapunov equation yields a well known Fredholm equation. The self-focused and accelerated particle beams are studied using an analytic solution to the self-consistent Vlasov equation. A Lorentz force is treated as a control parameter (a control vector describing control fields), where a problem of optimal control is resolved. According to the Poisson equations charge and current densities are studied in the framework of a programming problem in an usual form.

One of the aims is to study the oscillatory behavior of a solution to the Vlasov-Poisson equations, and dispersion properties of nonlinear waves.

MAIN RESULT

A sufficiently general kinetic description of a particle beam behavior in an electrostatic field is given by the Vlasov – Poisson system (VPS):

$$\begin{aligned} \partial_t f + v\partial_x f + E\partial_v f &= 0, \\ \nabla E &= -4\pi\rho. \end{aligned}$$

Here $f = f(t,x,v)$ is a distribution of particles in a phase space $\{x,v\}$ depending, on the time t; $x \in R^3$, $v \in R^3$, $E = E(t,x)$ is the electric field, and

$$\rho = q\int f(t,x,v)dv, \quad j = q\int v f(t,x,v)dv$$

are the charge density and the current density respectively.

The VPS problem consists in proving the existence of a C^1 or L^p solution $f(t,x,v)$ for all $t \geq 0$ where $f(0,x,v) = \xi(x,v)$ is a given function.

The report goal is to show a new approach for the numerical simulation of beam dynamics based on the universality of Maxwell equations (V.I. Zubov [1]) and L-moment problem (M.G. Krein [4]). Here we only briefly describe how the L-problem reduces questions of choice of needed field $E(x)$ to an approximation problem.

Let us represent the solution f as a sum

$$f = \sum C_k \psi_k e^{\lambda_k t}, \quad k = 1, 2, ...$$

where $\psi_k = \psi_k(x,v)$, C_k and λ_k are some constants.

This reasoning yields a simple equation

$$\lambda_k\psi_k + E\partial_v\psi_k = \int_{\Omega_x}\int_{\Omega_v} v\partial_x\Phi(x-y; v-u)\psi_k(y,u)dudy$$

where $\Phi(x-y; v-u)$ is a Chezaro kernel. The calculation can be carried out and a solution will be a unique one iff

$$\int_{\Omega_x}\int_{\Omega_v} E\partial_v\psi_k \cdot \psi_k^* dudy = 0. \qquad (1)$$

* zvf@compuserv.com.ua

Here ψ_k^* is a conjugate function to ψ_k, and Ω_x, Ω_v are space and velocity volumes where the motion is carried out.

Now we will develop the object in view some simple situation, somehow, in order to construct the $E(x)$ we can use relation (1). L-moment problem for a continuous medium of the physical system is considered at defined mesh points $\{\Theta_\alpha\}, \alpha = 1, 2, ..., M$ and may be written down as follows

$$\sum_{\alpha=1}^{N} E(\Theta_\alpha)\eta_k(\Theta_\alpha) = 0, \qquad k = 1, 2, ..., N,$$

$$\sum_{\alpha=1}^{N} E(\Theta_\alpha)\Omega_\alpha = 1$$

Here $\{\Theta_\alpha\}$ is sets of the special given random points, $N \leq \infty$, $\eta_k(\Theta_\alpha) = E\partial_v f_0$, and f_0 is some optimal process, such that

$$\partial_t = 0 \Rightarrow \partial_x f_0 = -\partial_v f_0.$$

This technique allows to provide a precise numerical calculation of the dynamics of charged particles in beams. The L-moment method allows studying the detailed characteristics of bunched beams, taking into account a distribution of particles, real self and external fields, construct optimal fields, and others.

It is obvious that function the $E(x)$ is such that the known equations are valid.

More precisely, the function $E(x)$ given above must be such that the following conditions hold true

$$\nabla E(x) = 4\pi q \int f(t, x, v)dv, \tag{2}$$

$$\mathrm{rot}E(x) = 0. \tag{3}$$

This system is overdetermined and for this reason we consider the following approach [5].

The second equation of the system (2), (3) is unresolved for any vector field E. Indeed, let $\mathrm{rot}E = \varpi$, $\varpi \in L^2$, $\int f dv \in L^2$ and from here we have got the following condition: $\mathrm{divrot}E = 0$. Thus we go to the equation $\mathrm{div}\varpi = 0$. But all fields in L^2 form a subspace $S \subset L^2$. Thus the system (2), (3) may be resolved in the subspace $S \times L^2$ of the space $L^2 \times L^2$ only. But an orthogonal supplement to subspace S in L^2 is the gradient functions which equal vanish on the boundary $\partial\Omega$. In this connection we shall associate some scalar function P. This reasoning yields the following system

$$\begin{array}{rl} \nabla E(x) & = 4\pi q \int f(t, x, v)dv, \\ \mathrm{rot}E(x) + \mathrm{grad}P & = 0 \end{array} \tag{4}$$

with condition $P_{\partial\Omega} = 0$ and other condition on the boundary $\partial\Omega_x : \beta E_{\partial\Omega_x} = \alpha$. The system (4) is the elliptic system and it can be resolved [5].

EXAMPLE

We consider equation

$$\partial_t f(t, x, v) + v\partial_x f(t, x, v) + E(x)\partial_v f(t, x, v) = 0 \tag{5}$$

where $x, v \in R^1$. Now we will be able to find a solution of (5) in a form $f = \sum C_k \Psi_k(x, v)e^{i\omega t}$ [6].

It leads to the equation

$$i\omega\Psi_k + v\partial_x\Psi_k + E\partial_v\Psi_k = 0. \tag{6}$$

The function $\Psi_k(x, v)$ can de represented in the form

$$\Psi_k = \int \sum_{k_1,k_2}^{N} \left(1 - \frac{|k_1|}{1+N}\right)\left(1 - \frac{|k_2|}{1+N}\right) \times$$

$$\times e^{in_1x+in_2v}e^{in_1y+in_2u}\Psi_k(y, u)d\mu$$

where $d\mu = dydu$.

Obviously, (6) is equivalent to

$$(i\omega + ik_1v)\Psi_k = R\,(f, E) \tag{7}$$

where

$$R\,(f, E) = \int \sum_{k_1,k_2}^{N} \left(1 - \frac{|k_1|}{1+N}\right)\left(1 - \frac{|k_2|}{1+N}\right) \times$$

$$\times ik_2E(x)e^{in_1x+in_2v}e^{in_1y+in_2u}\Psi_k(y, u)d\mu.$$

The equation (7) has a solution if and only if ω and k correspond to a dispersion function $\varepsilon\,(\omega, k)$.

The function Ψ_k will be the solution of (7) which is classic solution or weak solution. But the spectrum of the operator R is determined in accordance with the dispersion law.

In order to expound the paper clearly some notations will be nedeed. Let us introduce the following ones

$$(1 - \frac{|k|}{1+N}) = \mu_k,$$

$$\varphi_k(x.v) = \mu_{k_1}\mu_{k_2}e^{ik_1x+k_2v}.$$

For simplicity, we assume $x, v \in R^1$ and $v \ll c$.

Now we can rewrite (7) as follows

$$(i\omega - ik_1\mu_k v)\Psi(x, v) =$$

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\sum ik_2E(x)\varphi_k(x.v)\varphi_k^*(y, u)\Psi(y, u)d\tau \tag{8}$$

where $d\tau = dxdv,\ x, y \in [-\pi, \pi],\ v, u \in [-\pi, \pi]$.

By integrating (8), it is easy to see that for the first N values we have got

$$\Psi_N = (\omega - kv)^{-1}\sum_{1}^{N} k_2E(x)\varphi_k(x.v)h_k. \tag{9}$$

$$h_\alpha = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\varphi_\alpha^*(y, u)\Psi(y, u)dydu.$$

Using the notation

$$\phi_{ij} = \int\int E(y)\varphi_i(y,u)\varphi_j^*(y,u)dydu \qquad (10)$$

we can get at once that h_α satisfies the following equation

$$h_\alpha = (\omega - \alpha v)^{-1}\sum_1^N \phi_{\alpha q} h_q. \qquad (11)$$

Obviously if the electrostatic field E is given then we can find the matrix $\Phi_N = [\phi_{ij}]_1^N$ and will resolve the equation (11).

Proceeding as before, we have got the spectrum $\sigma(\Phi_N)$ of the matrix Φ_N. If $\lambda \in \sigma(\Phi_N)$ then we can find dispersion relations

$$\lambda = \omega - \alpha v, \qquad (\alpha = 1, ..., N).$$

On the other hand, if the vector-function E is unknown we have other conditions. It is easy to see that the matrix Φ_N is the Toeplitz one, and if we know its from other condition then the electrostatic field E may be calculated.

As it is known, a distribution of eigenvalues of the Toepliz matrix T is determined by the function g which generates $T(g)$.

In our case it is the function E. If it is a bounded function then there exist constants $-\infty < m,\, M < +\infty$ such that $m \leq \lambda_{\min},\ \lambda_{\max} \leq M$. Moreover, for any fixed N the following relation holds:

$$\lambda_\nu^N \ \div\ E\left(-\pi + \frac{2\nu\pi}{N+2}\right).$$

When taking for example a case

$$E(x) = E_0 + 2\left(a\cos x + d\sin x\right), \quad x \in [-\pi, \pi]$$

we obtain

$$\lambda_\nu^N = -2\cos\frac{\nu\pi}{N+2}, \quad \nu = 1, 2, ..., N+1.$$

If the function E is a Lebesgue function then the distribution $\sigma(T)$ coincides with the function $g(x)$.

ENDNOTE

In this paper we propose a new approach scheme for solving Vlasov-Maxwell problems on the base of control theory. From a mathematical point of view this means that the solutions of the nonlinear evolutionary wave equations have got simplification in the description by use any results of control algorithms.

This is due in particular to the presence of the universality of the Maxwell equation. A lot of analytical investigations and computer experiments are devoted to the study of this equation. Our earlier works are devoted to these investigations and also include, in some cases, new analytical results. In this report we briefly present an approach to study some problems of beams acceleration and strong focusing.

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