NONLINEAR THEORY OF EXCITATION OF AN AXIALLY ASYMMETRIC WAKEFIELD IN DIELECTRIC RESONATOR*

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Abstract

A nonlinear self-consistent theory of excitation of an axially asymmetric wakefield by relativistic electron bunches in cylindrical dielectric resonator with a vacuum channel for the charged particles transportation through the resonator is constructed. The formulated nonlinear theory allows investigating numerically the nonlinear effects such as increasing of the transverse bunch size, and head-tail beam breakup instability, which occurs if an electron bunch in the structure is misaligned.

INTRODUCTION

The dielectric wakefield accelerator is one of the modern trends of acceleration schemes, which can provide high-accelerating gradient for future colliders. But besides for high output energy of an accelerated bunches high demands are made on their quality, the same, for example, as low emittance. No loss of current under acceleration of the bunch are also desirable. This information about the bunch can not be obtained using assumption of the absence of reverse influence the excited field on the dynamics of electron bunches. In this paper we present nonlinear self-consistent theory of wakefield excitation in a dielectric–lined resonator by an electron bunches. The previous theoretical investigations on wakefield excitation in dielectric–lined structures, have been done for longitudinally unbounded structures [1]–[4]. In cited papers was noted, that it is necessary to taking into account the contribution of higher multipole modes to the total transverse field. A presented complete bunch–excited electromagnetic field includes all azimuthal modes, which allows calculating transverse wakefield in order to investigate bunch deflection problems.

STATEMENT OF THE PROBLEM

Consider cylindrical metallic resonator with inner radius b, partially filled with isotropic material with dielectric constant ε, containing on-axis vacuum channel of radius a which allows charged particles to pass through. We suppose that the end walls of the resonator are closed by metal grids transparent for charged particles and nontransparent for an excited electromagnetic field. Consider an electron bunch, injected into the resonator and moving along a line parallel to the axis of the resonator.

The electron bunches will be described in terms of macroparticles, therefore the charge density ρ and the current density $\mathbf{j}$ will be written as:

$$\rho = \sum_{p\in V_R} q_p \delta\left(\mathbf{r}-\mathbf{r}_p(t)\right),\ \mathbf{j} = \sum_{p\in V_R} q_p \mathbf{v}_p(t)\delta\left(\mathbf{r}-\mathbf{r}_p(t)\right), \qquad (1)$$

where q_p is the charge of the macroparticle, $\mathbf{r}_p$ and $\mathbf{v}_p$ are its time-dependent coordinates and velocity, respectively. The summation in Eq. (1) is carried out over the particles being in the resonator volume V_R.

FIELD SOLUTION

We introduce the solenoidal $\mathbf{E}^t$ $\mathbf{H}^t$ and the potential $\mathbf{E}^l = -\nabla\Phi$ fields defined as

$$\mathrm{div}(\varepsilon\mathbf{E}^t) = 0,\ \mathrm{div}(\mu\mathbf{H}^t) = 0,\ rot\mathbf{E}^l = 0, \qquad (2)$$

which are given by Maxwell's and Poisson equations:

$$rot\mathbf{E}^t = -\frac{\mu}{c}\frac{\partial\mathbf{H}^t}{\partial t},\ rot\mathbf{H}^t = \frac{\varepsilon}{c}\frac{\partial\mathbf{E}^t}{\partial t} + \frac{4\pi}{c}\mathbf{j}, \qquad (3)$$

$$\Delta(\varepsilon\Phi) = -4\pi\rho \qquad (4)$$

The solenoidal $\mathbf{E}^t$ and potential $\mathbf{E}^l$ electric fields are mutually orthogonal [5] and satisfy the boundary conditions, making their tangential components vanish on the metal walls of the resonator.

First we solve the equation (4) for the potential in the vacuum channel and dielectric. In cylindrical coordinate Eq.(4) rewrites as:

$$\frac{1}{\varepsilon r}\frac{\partial}{\partial r}\left(r\varepsilon\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\varphi^2} + \frac{\partial^2\Phi}{\partial z^2} = -\frac{4\pi}{\varepsilon}\rho \qquad (5)$$

Eq.(5) should be complemented by boundary conditions consisting in that the potential Φ on the resonator metal walls becomes zero

$$\Phi(z=0) = \Phi(z=L) = \Phi(r=b) = 0, \qquad (6)$$

*The research is supported in part by STCU, project №. P522.
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ISBN 978-3-95450-125-0

and continuity of the potential and radial component electric induction vector

$$\Phi(r=a-0)=\Phi(r=a+0),\ \left.\frac{\partial\Phi}{\partial r}\right|_{r=a-0}=\varepsilon\left.\frac{\partial\Phi}{\partial r}\right|_{r=a+0.} \quad (7)$$

By using the expansion by eigenfunctions method Eq. (5), with boundary conditions (6) and (7), can be solved. Finally, we obtain the potential in the form

$$\Phi(\mathbf{r},t)=\sum_{m=0}\sum_{n=1}\sum_{l=1}\frac{4\beta_m R_{mn}(r)\sin k_l z}{L\left(k_l^2+\kappa_{mn}^2\right)\|R_{mn}\|^2}\times$$
$$\times\sum_{p\in V_R} q_p R_{mn}(r_p)\cos m\left(\varphi-\varphi_p\right)\sin k_l z_p. \quad (8)$$

In the above (and below in paper) n, m, and l enumerate, respectively, radial, azimuthally and longitudinal indexes. Radial eigenfunctions $R_{mn}(r)$ and their norm have the form:

$$R_{mn}(r)=\begin{cases}\mathrm{J}_m(\kappa_{mn}r),\ 0\le r<a\\ \mathrm{J}_m(\kappa_{mn}a)\mathrm{Z}_m(\kappa_{mn}r)/\mathrm{Z}_m(\kappa_{mn}a),\ \ a\le r\le b\end{cases} \quad (9)$$

$$\|R_{mn}\|^2=\frac{a^2(1-\varepsilon)}{2}\left(1-\frac{m^2}{\kappa_{mn}^2a^2}\right)\mathrm{J}_m^2(\kappa_{mn}a)+$$
$$\varepsilon\frac{\mathrm{J}_m^2(\kappa_{mn}a)}{\mathrm{Z}_m^2(\kappa_{mn}a)}\frac{2}{\pi^2\kappa_{mn}^2}\frac{1}{\mathrm{Y}_m^2(\kappa_{mn}a)}+\frac{a^2}{2}\left(\mathrm{J}_m'(\kappa_{mn}a)\right)^2\left(1-\frac{1}{\varepsilon}\right), \quad (10)$$

where $\mathrm{Z}_m(\kappa r)\equiv\mathrm{J}_m(\kappa r)-\mathrm{J}_m(\kappa b)\mathrm{Y}_m(\kappa r)/\mathrm{Y}_m(\kappa b)$, J_m and Y_m are , respectively, Bessel function and Neumann function of order m; $k_l=\pi l/L,\ (l=0,1,...)$ are the longitudinal eigenvalues; $\beta_{m=0}=1,\ \beta_{m\neq 0}=2$. Radial eigenvalues κ_{mn} satisfies the equation

$$\varepsilon\mathrm{J}_m(\kappa a)\mathrm{Z}_m'(\kappa a)=\mathrm{Z}_m(\kappa a)\mathrm{J}_m'(\kappa a), \quad (11)$$

and can be found numerically.

The solenoidal parts of the electromagnetic field can be determined by expanding the required fields into solenoidal fields of the empty dielectric resonator [5]. Let us write down the fields $\mathbf{E}^{\mathbf{t}}$ and $\mathbf{H}^{\mathbf{t}}$ in the form:

$$\mathbf{E}^{\mathbf{t}}=\sum_s A_s(t)\mathbf{E}_s(\mathbf{r}),\quad \mathbf{H}^{\mathbf{t}}=-i\sum_s B_s(t)\mathbf{H}_s(\mathbf{r}). \quad (12)$$

The functions $\mathbf{E}_s$ and $\mathbf{H}_s$, which describe the spatial structure of solenoidal fields, satisfy the Maxwell sources-free equations.

By using the orthonormality conditions of eigenwaves

$$\int_{V_R}\varepsilon\mathbf{E}_s\mathbf{E}_{s'}^*dV=\int_{V_R}\mu\mathbf{H}_s\mathbf{H}_{s'}^*dV=4\pi N_s\delta_{ss'} \quad (13)$$

one can obtain the differential equations for calculation the expansion coefficients $A_s(t)$ and $B_s(t)$

$$\frac{d^2A_s}{dt^2}+\omega_s^2A_s=-\frac{dR_s}{dt},\ \frac{d^2B_s}{dt^2}+\omega_s^2B_s=-\omega_sR_s, \quad (14)$$

where $R_s=\dfrac{1}{N_s}\sum_{p\in V_R}q_p\mathbf{v}_p(t)\mathbf{E}_s^*[\mathbf{r}_p(t)]$.

Eigenfields, which satisfy the source–free Maxwell equations and electromagnetic boundary conditions can be written as:

$$\begin{cases}E_{r,s}=e_{r,s}(r)e^{im\varphi}\sin k_lz,\\ E_{\varphi,s}=-ie_{\varphi,s}(r)e^{im\varphi}\sin k_lz,\\ E_{z,s}=e_{z,s}(r)e^{im\varphi}\cos k_lz,\end{cases}\quad \begin{cases}H_{r,s}=h_{r,s}(r)e^{im\varphi}\cos k_lz,\\ H_{\varphi,s}=ih_{\varphi,s}(r)e^{im\varphi}\cos k_lz,\\ H_{z,s}=h_{z,s}(r)e^{im\varphi}\sin k_lz,\end{cases} \quad (15)$$

Then function R_s transforms to the expression

$$R_s=\frac{1}{N_s}\sum_{p\in V_R}q_p\Big(v_{pr}e_{r,s}(r_p)\sin k_lz_p\cos m\varphi_p+$$
$$v_{p\varphi}e_{\varphi,s}(r_p)\sin k_lz_p\sin m\varphi_p+v_{pz}e_{z,s}(r_p)\cos k_lz_p\cos m\varphi_p-$$
$$i\Big[v_{pr}e_{r,s}(r_p)\sin k_lz_p\sin m\varphi_p+v_{p\varphi}e_{\varphi,s}(r_p)\sin k_lz_p\cos m\varphi_p+$$
$$v_{pz}e_{z,s}(r_p)\cos k_lz_p\sin m\varphi_p\Big]\Big). \quad (16)$$

The functions describing the transverse structure of the solenoidal fields have the form:

$$\begin{aligned}e_{r,s}(r)&=\frac{k_s}{k_l^2-\varepsilon k_s^2}\left(\frac{m}{r}h_{z,s}+\frac{k_l}{k_s}\frac{de_{z,s}}{dr}\right),\\ e_{\varphi,s}(r)&=i\frac{k_s}{k_l^2-\varepsilon k_s^2}\left(\frac{k_l}{k_s}\frac{m}{r}e_{z,s}+\frac{dh_{z,s}}{dr}\right),\\ h_{r,s}(r)&=\frac{\varepsilon k_s}{\varepsilon k_s^2-k_l^2}\left(\frac{m}{r}e_{z,s}+\frac{k_l}{\varepsilon k_s}\frac{dh_{z,s}}{dr}\right),\\ h_{\varphi,s}(r)&=i\frac{\varepsilon k_s}{\varepsilon k_s^2-k_l^2}\left(\frac{k_l}{\varepsilon k_s}\frac{m}{r}h_{z,s}+\frac{de_{z,s}}{dr}\right),\end{aligned} \quad (17)$$

where transverse structure of the axial components $e_{z,s}(r)$ and $h_{z,s}(r)$, through which all others components can be expressed, defines as follows:

$$\begin{aligned}e_{z,s}(r)&=\begin{cases}\mathrm{J_m}(k_{v,s}r)/\mathrm{J_m}(k_{v,s}a),0\le r<a\\ \mathrm{Z_m}(k_{d,s}r)/\mathrm{Z_m}(k_{d,s}a),\ a<r\le b\end{cases}\\ h_{z,s}(r)&=\begin{cases}C_s\mathrm{J_m}(k_{v,s}r)/\mathrm{J_m}(k_{v,s}a),\ 0\le r<a\\ C_s\Phi_\mathrm{m}(k_{d,s}r)/\Phi_\mathrm{m}(k_{d,s}a),\ a<r\le b,\end{cases}\end{aligned} \quad (18)$$

where

$$C_s=-\frac{k_s^2k_lm(\varepsilon-1)}{k_{v,s}^2k_{d,s}^2aD_s},\ D_s=\frac{k_s}{k_{v,s}}\frac{\mathrm{J}_m'(k_{v,s}a)}{\mathrm{J}_m(k_{v,s}a)}-\frac{k_s}{k_{d,s}}\frac{\Phi_m'(k_{d,s}a)}{\Phi_m(k_{d,s}a)},$$

Eigenfrequencies ω_s are determined from the dispersion equation

$$\left(\frac{1}{k_v^2}-\frac{1}{k_d^2}\right)^2\frac{k_l^2m^2}{a^2}-\left[\frac{k}{k_v}\frac{J'_m(k_va)}{J_m(k_va)}-\frac{k}{k_d}\frac{\Phi'_m(k_da)}{\Phi_m(k_da)}\right]\times$$
$$\left[\frac{k}{k_v}\frac{J'_m(k_va)}{J_m(k_va)}-\frac{\varepsilon k}{k_d}\frac{Z'_m(k_da)}{Z_m(k_da)}\right]=0, \qquad (19)$$

where $k_s=\omega_s/c$ are the wave numbers, $k_{v,s}^2=\omega_s^2/c^2-k_l^2$, $k_{d,s}^2=\varepsilon\omega_s^2/c^2-k_l^2$ are the transverse wave numbers , respectively , in vacuum channel and in the dielectric; $Z_m(k_dr)\equiv J_m(k_dr)-J_m(k_db)Y_m(k_dr)/Y_m(k_db)$, $\Phi_m(k_dr)\equiv J_m(k_dr)-J'_m(k_db)Y_m(k_dr)/Y'_m(k_db)$.

Taking into account the expressions for the transverse structure of the solenoidal fields (17) and (18) we can write down the expressions for the norms N_s:

$$N_s=\frac{ak_s^2(\varepsilon-1)}{k_{v,s}^3k_{d,s}^2}\frac{J'_m(k_{v,s}a)}{J_m(k_{v,s}a)}\left(k_s^2C_s^2+k_l^2\right)+$$
$$\frac{k_s^2a^2}{2k_{v,s}^2}(1+C_s^2)\left[\left(1-\frac{m^2}{k_{v,s}^2a^2}\right)+\left(\frac{J'_m(k_{v,s}a)}{J_m(k_{v,s}a)}\right)^2\right]+$$
$$\frac{k_s^2\varepsilon C_s^2}{k_{d,s}^2}\left[\frac{2}{\pi^2k_{d,s}^2}\left(1-\frac{m^2}{k_{d,s}^2b^2}\right)\frac{1}{\left(\Phi_m(k_{d,s}a)Y'_m(k_{d,s}b)\right)^2}-\right.$$
$$\left.\frac{a^2}{2}\left\{\left(1-\frac{m^2}{k_{d,s}^2a^2}\right)+\left(\frac{\Phi'_m(k_{d,s}a)}{\Phi_m(k_{d,s}a)}\right)^2\right\}\right]+$$
$$\frac{k_s^2\varepsilon^2}{k_{d,s}^2}\left[\frac{2}{\pi^2k_{d,s}^2}\frac{1}{\left(Z_m(k_{d,s}a)Y_m(k_{d,s}b)\right)^2}-\right.$$
$$\left.\frac{a^2}{2}\left\{\left(1-\frac{m^2}{k_{d,s}^2a^2}\right)+\left(\frac{Z'_m(k_{d,s}a)}{Z_m(k_{d,s}a)}\right)^2\right\}\right]-$$
$$2k_sk_lmC_s\left(\frac{\varepsilon}{k_{d,s}^4}-\frac{1}{k_{v,s}^4}\right), \qquad (20)$$

For all of the components of the solenoidal electric and magnetic field the results are:

$$E_r(\mathbf{r},t)=\sum_{m=0}\sum_{n=1}\sum_{l=1}\beta_me_{r,s}(r)\sin k_lz\times$$
$$(\mathrm{Re}\,A_s(t)\cos m\varphi-\mathrm{Im}\,A_s(t)\sin m\varphi)$$
$$E_\varphi(\mathbf{r},t)=-\sum_{m=0}\sum_{n=1}\sum_{l=0}\beta_me_{\varphi,s}(r)\sin k_lz\times$$
$$(\mathrm{Im}\,A_s(t)\cos m\varphi+\mathrm{Re}\,A_s(t)\sin m\varphi) \qquad (21)$$
$$E_z(\mathbf{r},t)=\sum_{m=0}\sum_{n=1}\sum_{l=0}\beta_me_{z,s}(r)\cos k_lz\times$$
$$(\mathrm{Re}\,A_s(t)\cos m\varphi-\mathrm{Im}\,A_s(t)\sin m\varphi)$$

$$H_r(\mathbf{r},t)=\sum_{m=0}\sum_{n=1}\sum_{l=0}\beta_mh_{r,s}(r)\cos k_lz\times$$
$$\left(\mathrm{Im}\,B_s(t)\cos m\varphi+\mathrm{Re}\,B_s(t)\sin m\varphi\right)$$
$$H_\varphi(\mathbf{r},t)=\sum_{m=0}\sum_{n=1}\sum_{l=0}\beta_mh_{\varphi,s}(r)\cos k_lz\times$$
$$\left(\mathrm{Re}\,B_s(t)\cos m\varphi-\mathrm{Im}\,B_s(t)\sin m\varphi\right) \qquad (22)$$
$$H_z(\mathbf{r},t)=\sum_{m=0}\sum_{n=1}\sum_{l=1}\beta_mh_{z,s}(r)\sin k_lz\times$$
$$\left(\mathrm{Im}\,B_s(t)\cos m\varphi+\mathrm{Re}\,B_s(t)\sin m\varphi\right)$$

The self-consistent dynamics of bunch particles is described by relativistic equations of motion in the electromagnetic fields excited by bunches:

$$\frac{d\mathbf{p}_p}{dt}=q_p\left(\mathbf{E}+\frac{1}{m_pc\gamma_p}\mathbf{p}_p\times\mathbf{B}\right),\qquad \frac{d\mathbf{r}_p}{dt}=\frac{\mathbf{p}_p}{m_p\gamma_p}, \qquad (23)$$

where $\gamma_p^2=1+\left(\mathbf{p}_p/m_pc\right)^2$.

CONCLUSIONS

In present work a system of self-consistent equations describing the dynamics of excitation both an azimutally uniform and nonuniform modes of wakefield, excited by relativistic electron bunches in a dielectric resonator, are obtained.

An bunch–excited fields are presented in the form of superposition solenoidal and potential fields. The solenoidal electromagnetic fields are presented by an expansion of the required fields into solenoidal fields of the empty dielectric resonator. The potential field is presented by the eigenfunction expansion method. The dispersion equation for determination of eigenfrequencies and the equation for eigenvalues are obtained, eigenwaves, eigenfunctions and their norms are found.

The analytical expressions of an excited fields, that take into account both longitudinal and transverse dynamics of bunch particles are derived.

Along with the equations of motion they provide a self-consistent description of the dynamics of generated fields and bunches.

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ISBN 978-3-95450-125-0