INVESTIGATION OF PHASE TRAJECTORIES OF PARTICLE MOTION IN A SYNCHROTRON NEAR THE NONLINEAR RESONANCE OF THIRD ORDER

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Abstract

The nonlinear resonance of third order plays an important role in the particle dynamics in circular accelerators, colliders and storage rings and is widely used for slow extraction of particles from synchrotrons. Consideration is carried out in the canonical variables X, Y which at a given accelerator azimuth are simply related to the angle and the deviation of the circulating particles relative to the equilibrium orbit. The problem is reduced to the construction of the phase trajectories, which are the curves of the third or fourth order and determine the type of the motion near the resonance under consideration. The construction of the phase trajectories is performed by the Klein's perturbation method. The influence on the particles dynamics of octupole component of the magnetic fields is investigated.

INTRODUCTION

The problems of nonlinear dynamics play an important role in different fields of modern physics such as elementary particle physics, nuclear physics, plasma physics, quantum electronics and, certanly, in the particle accelerators physics [1]. The nonlinear resonance excitation is wiedly used for particle extraction from circular accelerators [2], the nonlinear resonances action determines dynamical aperture of large circular accelerators and storage rings. One of the most important problems in nonlinear oscillations study is construction of the trajectories of representive points on phase plane and stable motion regions finding.

BASIC THEORETICAL STATEMEMTS

When considering a particle motion in a circular accelerators it is convenient to replace longitudinal coordinate s by so called generalized azimuth ϕ

$$\phi = 2\pi \frac{s}{R_0},$$

where $R_0 = \Pi/2\pi$, and Π - accelerator perimeter.

The equation of one-dimensional particle motion in an accelerator in the presence of a perturbation has the form

$$\frac{d^2x}{d\phi^2} + \nu_x^2 x = \epsilon F\left(\phi, x, \frac{dx}{d\phi}\right), \tag{1}$$

where x is the transverse displacement of the circulating particle with respect to the equilubrium orbit, ν_x is the betatron oscillation frequency, ϵ is small positive parameter, $\phi = \int \frac{ds}{\nu_x \beta(s)}$, $\beta(s)$ is the betatron function. The function $F\left(\phi, x, \frac{dx}{d\phi}\right)$ is periodic with respect to ϕ function with period equal to 2π. Taking the smallness of the perturbation into acount the solution of the equation (1) can represented in the form, that it has for the homogeneous equation, but now with the amplitude a and the phase ψ depending on the azimuth ϕ [3].

$$x = a(\phi)\cos(\nu_x\phi + \psi(\phi)). \tag{2}$$

The amplitude a and phase ψ are subjected to the following equations

$$\begin{aligned} \frac{da}{d\phi} &= A(a,\psi), \\ \frac{d\psi}{d\phi} &= \Psi(a,\psi). \end{aligned} \tag{3}$$

Canonical variables

For further analysis it is convenient to move to the new variables X and Y [4]

$$X = a\cos\psi, \qquad Y = -a\sin\psi. \tag{4}$$

At a given accelerator azimuth variables X, Y are connected in a simple way with the angle and the displacement of the circulating particle with respect to equilibrium orbit. In these variables equations (3) take the canonical form

$$\begin{aligned} \frac{dX}{d\phi} &= \frac{\partial H}{\partial Y}, \\ \frac{dY}{d\phi} &= -\frac{\partial H}{\partial X}, \end{aligned} \tag{5}$$

where $H(X,Y)$ is the Hamiltonian. Such change of the variables and use of the Hamiltonian allow to make a descriptive analysis of the particles motion on the phase plane (X,Y). Curves on which the particles move are determined by the equation $H(X,Y) = const$. From the conditions

$$\begin{aligned} \frac{dX}{d\phi} &= \frac{\partial H}{\partial Y} = 0, \\ \frac{dY}{d\phi} &= -\frac{\partial H}{\partial X} = 0, \end{aligned} \tag{6}$$

the postions of the specific points on the phase plane are determined.

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Resonance terms In a special case the nonlinear field F appearing in the equation (1) can be represented in the following way

$$F = x^l \cos(m\phi), \tag{7}$$

l, m – integer. The substitution of (2), (7) in equation (1) and subbsequent expansion of the function F into a Fourie series causes resonance terms appearance. Resonance condition takes the form

$$\nu_x = \frac{m}{l+1}. \tag{8}$$

THIRD ORDER RESONANCE

The third order resonance $\nu = q/3$ is exited by a suitable q harmonic of the quadratic (sextupole $l = 2$) magnetic field $F(\varphi, x, dx/d\varphi) = -A_2x^2 \cos q\varphi - Bx^3$. Here the constant component of qubic (octupole $l = 3$) magnetic field B playing an important role in circular accelerators is keeping.

In this case the Hamiltonian can be represented in a normal form [5] and can be written as [4].

$$H = \frac{A_2}{16}(Y^3 - 3X^2Y) + \frac{1}{2}\left(\nu_x - \frac{q}{3}\right)(X^2 + Y^2) + \frac{B}{2}(X^2 + Y^2)^2. \tag{9}$$

By analogy with high energy physics [6] for further analysis of the phase trajectories we introduce the new designations

$$\begin{aligned} s &= Y - \sqrt{3}X + \frac{2\sqrt{3}}{3}X_0, \\ t &= Y + \sqrt{3}X + \frac{2\sqrt{3}}{3}X_0, \\ u &= Y - \frac{\sqrt{3}}{3}X_0, \end{aligned} \tag{10}$$

where $X_0 = 8\sqrt{3}(\nu_x - q/3)/3A_2$, and in accordance with the sign of the tune shift $\delta = (\nu_x - q/3)$ it takes positive or negative value. Taking into account (10) one can cast (9) in the form

$$s \cdot t \cdot u + \frac{8B}{A_2}(X^2 + Y^2)^2 = \frac{16}{A_2}H - \frac{4}{9}X_0^3. \tag{11}$$

In general, this is the equation of the fourth order curve. In the absence of the constant component of the cubic nonlinearity of the magnetic field ($B = 0$) it defines a curve of the third order. Phase trajectories are given by the equation

$$s \cdot t \cdot u = \eta, \tag{12}$$

where $\eta = 16H/A_2 - 4X_0^3/9$ – constant. By analogy with electrostatics value η can be called charge [7]. Then the entire phase plane is divided into two areas: one carrying a positive charge of $\eta > 0$, and the other carrying a negative charge of $\eta < 0$ [9]. The boundary between these regions is separatrix

$$s \cdot t \cdot u = 0, \tag{13}$$

on separatrix $H = \frac{\sqrt{3}}{36}A_2X_0^3$. The solution of this equation is given by family of three straight lines:

$$s = 0, \quad t = 0, \quad u = 0, \tag{14}$$

forming the equilateral triangle at its intersection. The vertexes of this triangle determine the position of three unstable fixed points

$$\begin{array}{llll} 1) & X = 0, & Y = & -\frac{2\sqrt{3}}{3}X_0; \\ 2) & X = -X_0, & Y = & \frac{\sqrt{3}}{3}X_0; \\ 3) & X = X_0, & Y = & \frac{\sqrt{3}}{3}X_0. \end{array}$$

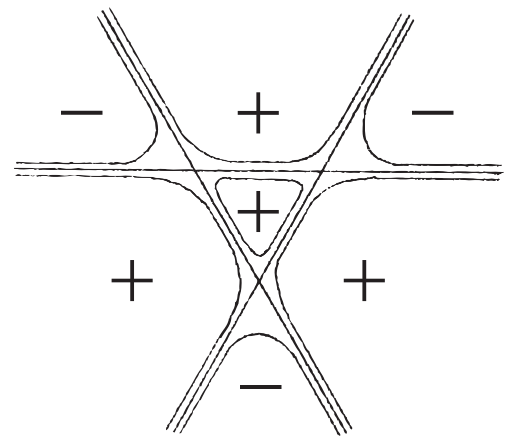

Figure 1: Schematic view of the phase trajectories near to the resonance of third order. Distribution of charge is shown. $B = 0$.

The separatrix divides the phase plane into regions of stable and unstable motion. Let us examine the nature of the phase curves. First of all it should be noted that each of the straight lines (14) divides the plane into two half-plane with positive or negative values of the corresponding variables s, t, u. According to this fact the sign of the $s \cdot t \cdot u$ product in (12) is determined. The phase trajectory bild-up may be performed in "the small variation method" explained by F. Klein [8]. For a small values of η, such that $|\eta| \ll |X_0^3|$ the corresponding phase trajectories are close to the separatrix in the region defined by the sign of η. It opens up a opportunity of graphical construction of phase trajectories on the basis of qualitative analysis [9] (fig. 3).

The phase curves farthest from the separatrix are close to the fixed points. In the central area there are the closed curves containing the stable singularity. If $X^2 + Y^2 = R^2 \gg X_0^2$ the phase curve distance from the separatrix decreases as η/R^2. When B is not equal to zero at the large distances from the center of the coordinate system ($R^2 \gg X_0^2$) term in (11) proportional to $(X^2 + Y^2)^2$ dominates and defines the sign of the left side of the equation keeping it constant everywhere outside of some central area with a large enough radius. This leads to the coalescence of the separatrix branches. Moreover there is the connection of the branches limiting the sectors with η opposite in the sign to B.

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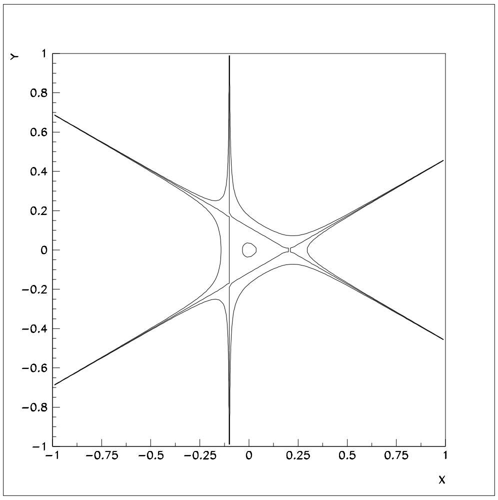

Figure 2: Phase trajectories close to the resonance of third order, $B = 0$.

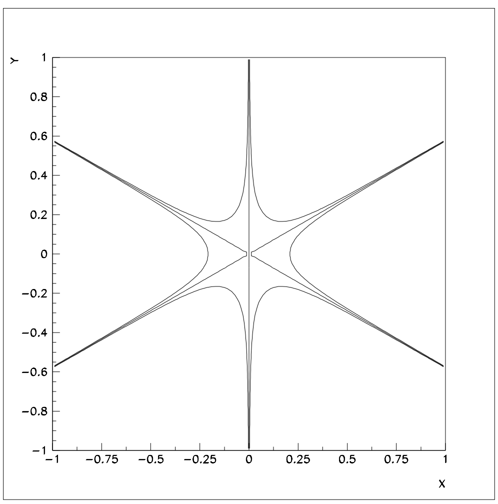

Figure 3: Phase trajectories at the resonance of third order, $B = 0$.

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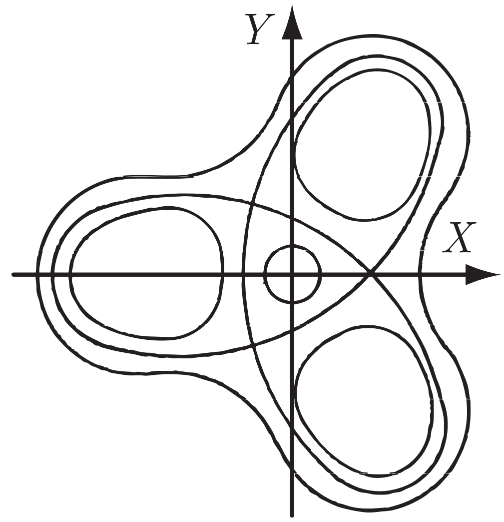

Figure 4: Phase trajectories close to the resonance of third order, $B \neq 0$.

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