THE MULTIPOLE LENS MATHEMATICAL MODELING

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Abstract

In the present work the mathematical model of the multipole system is presented. The multipole system is composed of arbitrary even number of the uniform electrodes. Each of the electrodes is a part of the plane. The potentials of the electrodes are the same modulus and opposite sign for neighboring electrodes. The variable separation method is used to solve the electrostatic problem. The potential distribution is represented as the eigen functions expansions. The boundary conditions and the normal derivative continuity conditions lead to the linear algebraic equations system relative to the series coefficients.

INTRODUCTION

Electrostatic multipole systems are widely used in the accelerator technology for the charged particle beams transport [1]– [3]. In this paper the mathematical modeling of the electrostatic multipole system is presented. The multipole system consist of the even number uniform plate electrodes of the same shape and size. Fig. 1 shows a schematic representation of the multipole system. The similar system was investigated in [4]. A quadrature expression was obtained for the field potential and the constraints imposed on the electrode potentials, under which such a solution is possible, were determined. In our work a system with an arbitrary even number $2N$ of electrodes is modeled. The variable separation method [5]– [7] is used in plane polar coordinates (r, α) to solve the boundary-value problem for the Laplace equation [8].

The multipole potential distribution has the planes of symmetry $\alpha = (\pi k)/N$ and planes of antisymmetry $\alpha = \pi/(2N) + (\pi k)/N$, $k = \overline{0, N-1}$. An additional plane $r = R_2$ can be introduce to limit the area of the problem under consideration without loss of generality. Thus it suffices to consider sector $0 \leq \alpha \leq \pi/2N$, $0 \leq r \leq R_2$ to find the electrostatic field. Schematic diagram of the multipole system sector is presented on Fig. 2 ($\alpha_1 = \pi/2N$).

The problem parameters are:

$(R_1, 0)$ — the coordinate position of the multipole electrode's edge,

R_2 — the radius of the area,

$\alpha_1 = \pi/2N$ — the boundary of the area (the plane of antisymmetry),

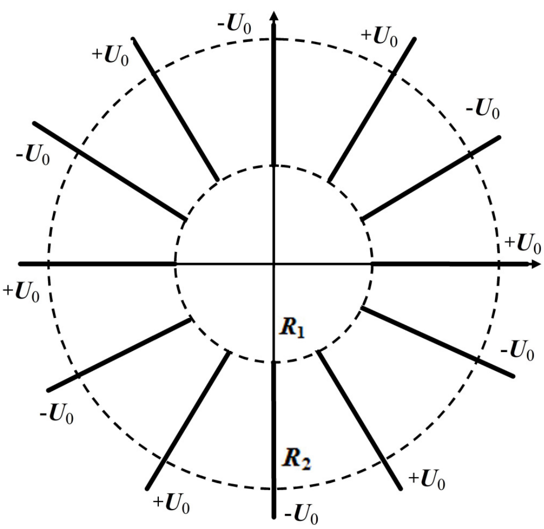

Figure 1: Schematic representation of the multipole system.

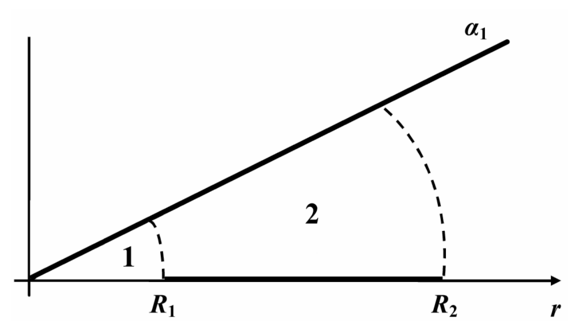

Figure 2: Schematic representation of the multipole system sector.

U_0 — the multipole electrode potential ($\alpha = 0$, $R_1 \leq r \leq R_2$).

MATHEMATICAL MODEL

The electrostatic potential distribution $U(r, \alpha)$ in the area $(0 \leq r \leq R_2, 0 \leq \alpha \leq \alpha_1)$ satisfies the Laplace equation and the boundary conditions

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$$\begin{aligned} &\triangle\, U(r,\alpha) = 0, \quad 0 \le r \le R_2,\ 0 \le \alpha \le \alpha_1; \\ &U(R_1, 0) = U_0, \quad R_1 \le r \le R_2; \\ &U(R_2, \alpha) = 0, \quad 0 \le \alpha \le \alpha_1; \\ &U(r, \alpha_1) = 0, \quad 0 \le r \le R_2; \\ &\left.\frac{\partial U}{\partial \alpha}\right|_{\alpha = 0} = 0, \quad 0 \le r \le R_1, \end{aligned} \tag{1}$$

where
$\alpha = 0$ — the plane of symmetry,
$\alpha = \alpha_1$ — the plane of antisymmetry.

SOLUTION OF THE BOUNDARY – VALUE PROBLEM

The variable separation method is used to solve the boundary-value problem (1). The interior of the multipole system sector domain ($0 \le r \le R_2$, $0 \le \alpha \le \alpha_1$) can be divided into two subdomains:

subdomain 1: ($0 \le \alpha \le \alpha_1, 0 \le r \le R_1$);

subdomain 2: ($0 \le \alpha \le \alpha_1, R_1 \le r \le R_2$).

The potential distribution function $U(r, \alpha)$ can be presented as

$$U(r,\alpha) = \\ = \begin{cases} U_1(r,\alpha), & 0 \le \alpha \le \alpha_1,\ 0 \le r \le R_1; \\ U_2(r,\alpha), & 0 \le \alpha \le \alpha_1,\ R_1 \le r \le R_2. \end{cases} \tag{2}$$

Then functions $U_1(r, \alpha)$, $U_2(r, \alpha)$ (2) are represented as a Fourier series:

$$U_1(r,\alpha) = \sum_{n=1}^{\infty} a_n \left(\frac{r}{R_1}\right)^{\mu_n} \cos(\mu_n \alpha), \tag{3}$$

$$\begin{aligned} U_2(r,\alpha) = &\sum_{k=1}^{\infty} b_k \frac{\left(\frac{r}{R_2}\right)^{\nu_k} - \left(\frac{R_2}{r}\right)^{\nu_k}}{\left(\frac{R_1}{R_2}\right)^{\nu_k} - \left(\frac{R_2}{R_1}\right)^{\nu_k}} \times \\ &\times \cos(\nu_k \alpha) + \\ &+ \sum_{m=1}^{\infty} c_m \frac{\sinh(\lambda_m(\alpha_1 - \alpha))}{\sinh(\lambda_m \alpha_1)} \sin\left(\lambda_m \ln \frac{r}{R_2}\right), \end{aligned} \tag{4}$$

where

$$\mu_n = \frac{\pi(2n+1)}{2\alpha_1}, \\ \nu_k = \frac{\pi k}{2\alpha_1}, \\ \lambda_m = \frac{\pi m}{\ln \frac{R_1}{R_2}}. \tag{5}$$

The potential distribution functions (2)–(5) are the Laplace equation solutions for each subdomain ($i = 1, 2$). The boundary conditions (1) are satisfied for $\alpha = \alpha_1$ ($0 \le r \le R_2$), $r = R_2$ ($0 \le \alpha \le \alpha_1$), $\alpha = 0$ ($0 \le r \le R_1$).

The coefficients c_m in expansion (4) can be calculated in an explicit form with the boundary conditions (1) $\alpha = 0$ ($R_1 \le r \le R_2$):

$$c_m = \frac{U_0}{\pi m}(1 - (-1)^m). \tag{6}$$

The continuity condition for the potential distribution $U_1(R_1, \alpha) = U_2(R_1, \alpha)$ for $0 \le \alpha \le \alpha_1$ leads to the relation between the coefficients a_n and b_k:

$$\begin{aligned} a_n &= \frac{2}{\alpha_1} \sum_{k=1}^{\infty} b_k \frac{\nu_k}{\nu_k^2 - \mu_n^2} = \\ &= \frac{8}{\pi} \sum_{k=1}^{\infty} b_k \frac{k}{4k^2 - (2n+1)^2}. \end{aligned} \tag{7}$$

The normal derivative of the electric displacement vector continuity conditions can be written as

$$\left.\frac{\partial U_1(r,\alpha)}{\partial r}\right|_{r \le R_1} = \left.\frac{\partial U_2(r,\alpha)}{\partial r}\right|_{r \le R_1}. \\ 0 \le \alpha \le \alpha_1. \tag{8}$$

Equation (8) establishes another relation between the coefficients a_n and b_k:

$$\begin{aligned} a_n = &\frac{2}{\alpha_1 \mu_n} \times \\ &\times \left[\sum_{k=1}^{\infty} b_k \frac{\nu_k^2}{\nu_k^2 - \mu_n^2} \frac{\left(\frac{R_2}{R_1}\right)^{\nu_k} + \left(\frac{R_1}{R_2}\right)^{\nu_k}}{\left(\frac{R_2}{R_1}\right)^{\nu_k} - \left(\frac{R_1}{R_2}\right)^{\nu_k}} + \right. \\ &\left. + \frac{U_0}{\ln \frac{R_1}{R_2}} \sum_{m=1}^{\infty} \frac{(1 - (-1)^m)\lambda_m}{\lambda_m^2 + \mu_n^2} \coth \lambda_m \alpha_1 \right]. \end{aligned} \tag{9}$$

Then, in consequence of the formulas (7) and (9) the coefficients b_k are the solutions of the linear algebraic equations system:

$$\begin{aligned} &\sum_{k=1}^{\infty} b_k \frac{\nu_k}{\nu_k^2 - \mu_n^2} \times \\ &\times \left[\frac{\nu_k}{\mu_n} \frac{\left(\frac{R_2}{R_1}\right)^{\nu_k} + \left(\frac{R_1}{R_2}\right)^{\nu_k}}{\left(\frac{R_2}{R_1}\right)^{\nu_k} - \left(\frac{R_1}{R_2}\right)^{\nu_k}} - 1 \right] = \\ &= \frac{U_0}{\mu_n \ln \frac{R_1}{R_2}} \sum_{m=1}^{\infty} \frac{(1 - (-1)^m)\lambda_m}{\lambda_m^2 + \mu_n^2} \coth \lambda_m \alpha_1. \end{aligned} \tag{10}$$

CONCLUSION

In this article a mathematical model of the electrostatic multipole system is considered. The multipole system electrodes are the the uniform plate electrodes of the same shape and size. The variable separation method is applied to calculate the electrostatic potential distribution for the boundary-value problem (1) in plane polar coordinates. The potential distribution function is represented as Fourier series (3)–(5). Some of the coefficients (6) are determined from the boundary conditions (1) in an explicit form. The continuity conditions for the potential distribution and normal derivative of the electric displacement vector continuity conditions makes it possible to reduce the original boundary value problem (1) to the system of the linear algebraic equations with the constant coefficients (7), (10). All geometric dimensions and potential of the multipole system electrodes are the parameters of the problem.

REFERENCES

[1] H. Liu et al., "Design and Optimization of Multipole Lens and Wien Filter Systems," Nucl. Instrum. Methods Phys. Res. A 645 (2011) 300.

[2] P.W.Hawkes, "The Correction of Electron Lens Aberrations," Ultramicroscopy 156 (2015) A1.

[3] H. Sawada, "Aberration Correction in STEM (Book Chapter)," *Scanning Transmission Electron Microscopy of Nanomaterials: Basics of Imaging and Analysis*, (2014) 283.

[4] I.F. Spivak-Lavrov, "Solution of the dirichlet problem for the Laplace equation for a multiply connected region with point symmetry," Technical Physics 44(3) (1999) 223.

[5] N.V. Egorov and E.M. Vinogradova, "Mathematical modeling of the electron beam formatting systems on the basis of field emission cathodes with various shapes," Vacuum 72 (2) (2004) 103.

[6] E.M. Vinogradova et al., "Mathematical simulation of a diode system with a cylindrical field-emission tip," Technical Physics 60(2) (2015) 176.

[7] E.M. Vinogradova et al., "Mathematical modeling of field emitter array," Vacuum 127 (2016) 45.

[8] E.M. Vinogradova and A.V. Listrukova, "The Electrostatic Quadrupole Lens Mathematical Modeling," International Conference on "Stability and Control Processes" in Memory of V.I. Zubov, SCP 2015, Saint Petersburg, October 2015, p. 214 (2015).

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