SYMMETRICAL PARAMETERIZATION FOR 6D FULLY COUPLED ONE-TURN TRANSPORT MATRIX*

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Abstract

Symmetry properties of 6D and 4D one-turn symplectic transport matrices were studied. A new parameterization was proposed for 6D matrix, which is an extension of the Lebedev—Bogacz parameterization for 4D case. The parameterization is fully symmetric relative to radial, vertical and longitudinal motion. It can be useful for lattices with strong coupling between all degrees of freedom.

INTRODUCTION

For the case of a 2×2 transport matrix a well-known Twiss parameterization exists [1]. It can be used also in the case of 4×4 and 6×6 matrices if there is no coupling between different degrees of freedom. The usual case is when transversal modes are uncoupled but there is a small interaction between either of them (or both) and longitudinal one. But if longitudinal tune is much smaller than transversal ones, then longitudinal Twiss functions are assumed to be constant. So, longitudinal motion is eliminated and taken into account only in terms of the dispersion functions.

If there is an interaction between transversal modes, then different parameterizations for coupled motion can be used [2], [3], [4]. These parameterizations make use of the fact that horizontal and vertical degrees of freedom are identical mathematically. But this is also the case for the longitudinal one. In this paper we will derive some symmetry properties of a 6×6 transport matrix and build up a totally symmetrical parameterization for it. This parameterization can be used for lattices with strong coupling between all degrees of freedom. Then using the same approach we will reduce the dimensionality to 4×4 and derive Lebedev—Bogacz parameterization [4].

BASIC DEFINITIONS

Let us define a *block-diagonal matrix* as matrix having non-zero elements only within its 2×2 diagonal blocks. Then we introduce the following notation

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \mathbf{S} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ \begin{matrix} \mathbf{I}_6 = \mathrm{diag}\,(\,\mathbf{I} \quad \mathbf{I} \quad \mathbf{I}\,) \\ \mathbf{S}_6 = \mathrm{diag}\,(\,\mathbf{S} \quad \mathbf{S} \quad \mathbf{S}\,) \end{matrix} \,.$$

Let $\mathbf{M}$ be a 6×6 symplectic one-turn transport matrix, i.e. $\mathbf{M}^T\mathbf{S}_6\mathbf{M} = \mathbf{S}_6$. Then all the eigenvalues of $\mathbf{M}$ can be grouped into mutually inverse pairs [1]. Let λ_1 and λ_2 form such a pair, $\mathbf{v}_1$ and $\mathbf{v}_2$ be their eigenvectors, then

$$\overset{\leftrightarrow}{\mathbf{M}}\,\mathbf{v}_{1,2} = \hat{\lambda}\mathbf{v}_{1,2}\,, \tag{1}$$

where $\overset{\leftrightarrow}{\mathbf{M}} = \mathbf{M} + \mathbf{M}^{-1}$ is *recurrent matrix*, and $\hat{\lambda} = \lambda_1 + \lambda_2$. This means that $\overset{\leftrightarrow}{\mathbf{M}}$ has at most 3 different eigenvalues, each of them is degenerated at least twice. $\mathbf{M}$ describes stable motion if and only if all $|\lambda_j| = 1$ [1], so $\hat{\lambda} = 2\,\mathrm{Re}\,\lambda_{1,2}$.

For any 2×2 matrix $\mathbf{A}$ a *pseudoinversed matrix* $\hat{\mathbf{A}}$ can be defined (this operation was introduced in [1] as "symplectic conjugate")

$$\hat{\mathbf{A}} = -\mathbf{S}\mathbf{A}^T\mathbf{S}$$

with the following properties

$$\mathbf{A} + \hat{\mathbf{A}} = (\mathrm{Tr}\,\mathbf{A})\mathbf{I}\,, \quad \mathbf{A}\hat{\mathbf{A}} = |\mathbf{A}|\mathbf{I}\,.$$

Now one can write down the 6×6 transport matrix, its inverse and recurrent matrix in a blockwise form

$$\mathbf{M} = \begin{pmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} & \mathbf{M}_{13} \\ \mathbf{M}_{21} & \mathbf{M}_{22} & \mathbf{M}_{23} \\ \mathbf{M}_{31} & \mathbf{M}_{32} & \mathbf{M}_{33} \end{pmatrix}, \ \overset{\leftrightarrow}{\mathbf{M}} = \begin{pmatrix} b_1\mathbf{I} & \mathbf{R}_3 & \hat{\mathbf{R}}_2 \\ \hat{\mathbf{R}}_3 & b_2\mathbf{I} & \mathbf{R}_1 \\ \mathbf{R}_2 & \hat{\mathbf{R}}_1 & b_3\mathbf{I} \end{pmatrix},$$

$$\mathbf{M}^{-1} = \begin{pmatrix} \hat{\mathbf{M}}_{11} & \hat{\mathbf{M}}_{21} & \hat{\mathbf{M}}_{31} \\ \hat{\mathbf{M}}_{12} & \hat{\mathbf{M}}_{22} & \hat{\mathbf{M}}_{32} \\ \hat{\mathbf{M}}_{13} & \hat{\mathbf{M}}_{23} & \hat{\mathbf{M}}_{33} \end{pmatrix},$$

where $\mathbf{R}_i = \mathbf{M}_{\overrightarrow{i}\overleftarrow{i}} + \hat{\mathbf{M}}_{\overleftarrow{i}\overrightarrow{i}}$, $b_i = \mathrm{Tr}\,\mathbf{M}_{ii}$ and $|\mathbf{R}_i| = d_i$. From now on we will assume that indices $i, j \in \{1, 2, 3\}$, also $\overleftarrow{i}$ and $\overrightarrow{i}$ mean cyclic permutation of these values (e.g. $1 = \overleftarrow{2} = \overrightarrow{3}$).

EIGENVALUES OF RECURRENT MATRIX

Our method for eigenvalues calculation is similar to the one proposed in [2]. Let $\overset{\leftrightarrow}{\mathbf{v}}$ be 6-component eigenvector of $\overset{\leftrightarrow}{\mathbf{M}}$, i.e. $\overset{\leftrightarrow}{\mathbf{M}}\overset{\leftrightarrow}{\mathbf{v}} = \hat{\lambda}\,\overset{\leftrightarrow}{\mathbf{v}}$. We split $\overset{\leftrightarrow}{\mathbf{v}}$ into 3 two-component subvectors, so as $\overset{\leftrightarrow}{\mathbf{v}}^T = (\,\mathbf{X}_1^T \quad \mathbf{X}_2^T \quad \mathbf{X}_3^T\,)^T$. Then

$$a_{ij}\mathbf{X}_i + \mathbf{R}_{\overleftarrow{i}}\mathbf{X}_{\overrightarrow{i}} + \hat{\mathbf{R}}_{\overrightarrow{i}}\mathbf{X}_{\overleftarrow{i}} = \bar{\mathbf{0}}\,, \tag{2}$$

where $\bar{\mathbf{0}}$ is a zero two-component vector, $a_{ij} = b_i - \hat{\lambda}_j$. Eliminating 2 of 3 $\mathbf{X}_i$ one can obtain

$$a_{1j}d_1 + a_{2j}d_2 + a_{3j}d_3 - a_{1j}a_{2j}a_{3j} = t\,, \tag{3}$$

where $t = \mathrm{Tr}\,(\mathbf{R}_1\mathbf{R}_2\mathbf{R}_3)$.

As we proved earlier, each eigenvalue of $\overset{\leftrightarrow}{\mathbf{M}}$ is degenerated at least twice, so its characteristic polynomial is a perfect square of some $\hat{P}(\hat{\lambda})$ with real coefficients. So, (3) can be regarded as characteristic equation of $\overset{\leftrightarrow}{\mathbf{M}}$, i.e.

$$\begin{aligned} \hat{P}(\hat{\lambda}) = \sqrt{|\,\overset{\leftrightarrow}{\mathbf{M}} - \hat{\lambda}\mathbf{I}|} = \hat{\lambda}^3 - (b_1 + b_2 + b_3)\hat{\lambda}^2 + \\ +(b_1b_2 + b_2b_3 + b_1b_3 - d_1 - d_2 - d_3)\hat{\lambda} + \\ +(b_1d_1 + b_2d_2 + b_3d_3 - b_1b_2b_3 - t) \end{aligned}$$

* This work has been supported by Russian Science Foundation (project N14-50-00080).

ISBN 978-3-95450-181-6

Let us also introduce the following notation

$$p_j = \hat{P}'(\hat{\lambda}_j) = (\hat{\lambda}_{\vec{j}} - \hat{\lambda}_j)(\hat{\lambda}_{\overset{\leftarrow}{j}} - \hat{\lambda}_j)\,.$$

The 3 roots of $\hat{P}(\hat{\lambda})$ can be found using Cardano formulae. Matrix $\mathbf{M}$ describes stable motion if and only if all of them are real and lie inside the region $(-2;2)$. If there are less than 3 different roots, then spectrum of $\mathbf{M}$ is degenerated, this case will not be covered in the present paper. It is important that in non-degenerated case $p_j \neq 0$.

EIGENVECTORS OF RECURRENT MATRIX

Let us reexamine the (2) system. It can be rewritten as 9 different equivalent equation pairs

$$u_{ij}\mathbf{X}_{\vec{i}} = \hat{\mathbf{W}}_{\overset{\leftarrow}{i}\,j}\mathbf{X}_i\,, \quad u_{ij}\mathbf{X}_{\overset{\leftarrow}{i}} = \mathbf{W}_{\vec{i}\,j}\mathbf{X}_i\,, \tag{4}$$

where

$$\mathbf{W}_{ij}=\frac{1}{p_j}\left(\hat{\mathbf{R}}_{\overset{\leftarrow}{i}}\hat{\mathbf{R}}_{\vec{i}} - a_{ij}\mathbf{R}_i\right),\ u_{ij}=\frac{1}{p_j}\left(a_{\vec{i}\,j}a_{\overset{\leftarrow}{i}\,j} - d_i\right).$$

Using (4) one can write down 3 different matrices whose columns are eigenvectors of $\overset{\leftrightarrow}{\mathbf{M}}$

$$\overset{\leftrightarrow}{\mathbf{W}}_j = \begin{pmatrix} u_{1j}\mathbf{I} & \mathbf{W}_{3\vec{j}} & \hat{\mathbf{W}}_{2\overset{\leftarrow}{j}} \\ \hat{\mathbf{W}}_{3j} & u_{2\vec{j}}\mathbf{I} & \mathbf{W}_{1\overset{\leftarrow}{j}} \\ \mathbf{W}_{2j} & \hat{\mathbf{W}}_{1\vec{j}} & u_{3\overset{\leftarrow}{j}}\mathbf{I} \end{pmatrix}.$$

The following properties of $\mathbf{W}_{ij}$ and u_{ij} can be found directly

$$\begin{gathered}\mathbf{W}_{ij}\mathbf{W}_{\vec{i}\,j} = u_{\overset{\leftarrow}{i}\,j}\hat{\mathbf{W}}_{\overset{\leftarrow}{i}\,j}, \quad |\mathbf{W}_{ij}| = u_{\vec{i}\,j}u_{\overset{\leftarrow}{i}\,j}, \\ \mathbf{W}_{i1} + \mathbf{W}_{i2} + \mathbf{W}_{i3} = \mathbf{0},\end{gathered} \tag{5}$$

$$u_{i1} + u_{i2} + u_{i3} = u_{1j} + u_{2j} + u_{3j} = 1\,, \tag{6}$$

then $\overset{\leftrightarrow}{\mathbf{W}}_1 + \overset{\leftrightarrow}{\mathbf{W}}_2 + \overset{\leftrightarrow}{\mathbf{W}}_3 = \mathbf{I}_6$.

One can combine u_{ij} into 3×3 *coupling matrix*. According to (6), it contains 4 independent parameters and can be parameterized with u_{11}, u_{22}, u_{33} and *coupling asymmetry* l, which is

$$l = u_{32} - u_{23} = u_{13} - u_{31} = u_{21} - u_{12}\,.$$

TWISS PARAMETERIZATION

Let $\mathbf{W}_j$ be 3 matrices whose columns are the eigenvectors of $\mathbf{M}$. System (1) means that eigenvector of $\mathbf{M}$ corresponding to eigenvalue λ is the linear combination of two eigenvectors of $\overset{\leftrightarrow}{\mathbf{M}}$ corresponding to eigenvalue $\hat{\lambda} = \lambda + \lambda^{-1}$, hence one can find such block-diagonal matrices $\mathbf{Q}_j$ that

$$\mathbf{W}_j = \overset{\leftrightarrow}{\mathbf{W}}_j\,\mathbf{Q}_j = \overset{\leftrightarrow}{\mathbf{W}}_j \cdot \mathrm{diag}\begin{pmatrix}\mathbf{Q}_{1j} & \mathbf{Q}_{2\vec{j}} & \mathbf{Q}_{3\overset{\leftarrow}{j}}\end{pmatrix}.$$

Therefore, the following matrices are also block-diagonal

$$\mathbf{T}_j = \overset{\leftrightarrow}{\mathbf{W}}_j{}^{-1}\mathbf{M}\,\overset{\leftrightarrow}{\mathbf{W}}_j = \mathrm{diag}\begin{pmatrix}\mathbf{T}_{1j} & \mathbf{T}_{2\vec{j}} & \mathbf{T}_{3\overset{\leftarrow}{j}}\end{pmatrix} \tag{7}$$

with $|\mathbf{T}_{ij}| = 1$. So, well-known Twiss parameterization can be introduced for these blocks [1]

$$\begin{gathered}\mathbf{T}_{ij} = \mathbf{I}\cos\mu_j + \mathbf{J}_{ij}\sin\mu_j\,, \quad \mathbf{J}_{ij} = \begin{pmatrix}\alpha_{ij} & \beta_{ij} \\ -\gamma_{ij} & -\alpha_{ij}\end{pmatrix}, \\ \gamma_{ij} = \tfrac{1+\alpha_{ij}{}^2}{\beta_{ij}}\,, \quad \mu_j = \arg\lambda_j\,.\end{gathered} \tag{8}$$

Note that since μ_j are fixed, then β_{ij} can be negative for $i \neq j$. Diagonal blocks of $\mathbf{Q}_j$ can be expressed as

$$\mathbf{Q}_{ij} = \frac{1}{\sqrt{2\beta_{ij}}}\begin{pmatrix}-\beta_{ij} & \beta_{ij} \\ \alpha_{ij} - I & -\alpha_{ij} - I\end{pmatrix},$$

where I is imaginary unit.

From (7) and (8) the following commutation rules can be obtained

$$\mathbf{W}_{ij}\mathbf{J}_{\overset{\leftarrow}{i}\,j} = \mathbf{J}_{\vec{i}\,j}\mathbf{W}_{ij}\,. \tag{9}$$

Then a closed expression for $\mathbf{M}$ can be found

$$\mathbf{M} = \overset{\leftrightarrow}{\mathbf{W}}_1\,\mathbf{T}_1 + \overset{\leftrightarrow}{\mathbf{W}}_2\,\mathbf{T}_2 + \overset{\leftrightarrow}{\mathbf{W}}_3\,\mathbf{T}_3\,. \tag{10}$$

Using (5), all $\mathbf{W}_{ij}$ can be expressed in terms of $\mathbf{W}_{11}$, $\mathbf{W}_{22}$, $\mathbf{W}_{33}$ and u_{ij} in 2 different ways

$$\begin{aligned} l\mathbf{W}_{j\overset{\leftarrow}{j}} &= u_{\vec{j}\overset{\leftarrow}{j}}\mathbf{W}_{jj} - \hat{\mathbf{W}}_{\overset{\leftarrow}{j}\overset{\leftarrow}{j}}\hat{\mathbf{W}}_{\vec{j}\vec{j}} \\ l\mathbf{W}_{j\vec{j}} &= -u_{\overset{\leftarrow}{j}\vec{j}}\mathbf{W}_{jj} + \hat{\mathbf{W}}_{\overset{\leftarrow}{j}\overset{\leftarrow}{j}}\hat{\mathbf{W}}_{\vec{j}\vec{j}} \end{aligned}\,, \tag{11}$$

or

$$\begin{aligned} \tau_j\mathbf{W}_{j\overset{\leftarrow}{j}} &= \mathbf{J}_{\vec{j}\overset{\leftarrow}{j}}\mathbf{A}_j + \mathbf{A}_j\mathbf{J}_{\overset{\leftarrow}{j}\overset{\leftarrow}{j}} \\ \tau_j\mathbf{W}_{j\vec{j}} &= \mathbf{J}_{\vec{j}\vec{j}}\mathbf{B}_j + \mathbf{B}_j\mathbf{J}_{\overset{\leftarrow}{j}\vec{j}} \\ \tau_j &= \mathrm{Tr}\,(\mathbf{J}_{\vec{j}\vec{j}}\mathbf{J}_{\vec{j}\overset{\leftarrow}{j}} - \mathbf{J}_{\overset{\leftarrow}{j}\vec{j}}\mathbf{J}_{\overset{\leftarrow}{j}\overset{\leftarrow}{j}}) \\ \mathbf{A}_j &= \mathbf{W}_{jj}\mathbf{J}_{\overset{\leftarrow}{j}\vec{j}} - \mathbf{J}_{\vec{j}\vec{j}}\mathbf{W}_{jj} \\ \mathbf{B}_j &= \mathbf{W}_{jj}\mathbf{J}_{\overset{\leftarrow}{j}\overset{\leftarrow}{j}} - \mathbf{J}_{\vec{j}\overset{\leftarrow}{j}}\mathbf{W}_{jj} \end{aligned}\,. \tag{12}$$

Knowing $|\mathbf{W}_{ij}|$ from (5), one can obtain the following expressions from (9)

$$\mathbf{W}_{ij} = r_{ij}(\mathbf{I}\cos\phi_j + \mathbf{J}_{\vec{i}\,j}\sin\phi_j)(\mathbf{J}_{\vec{i}\,j} + \mathbf{J}_{\overset{\leftarrow}{i}\,j})$$

where $r_{ij} = \sqrt{\dfrac{u_{\vec{i}\,j}u_{\overset{\leftarrow}{i}\,j}}{2 - \mathrm{Tr}\,(\mathbf{J}_{\vec{i}\,j}\mathbf{J}_{\overset{\leftarrow}{i}\,j})}}$.

ϕ_j can be expressed as follows

$$\phi_j = \arctan\frac{f_j g'_j - f'_j g_j}{h_j f'_j - h'_j f_j} + \left(\frac{1}{2} \pm \frac{1}{2}\right)\pi\,, \tag{13}$$

ISBN 978-3-95450-181-6

where

$$
\begin{array}{ll}
f_j = |\mathbf{C}''_j| - \tau_j^2 u_{\underset{j}{\rightarrow}\underset{j}{\leftarrow}} u_{\underset{j}{\leftarrow}\underset{j}{\leftarrow}} & f'_j = |\mathbf{D}''_j| - \tau_j^2 u_{\underset{j}{\rightarrow}\underset{j}{\rightarrow}} u_{\underset{j}{\leftarrow}\underset{j}{\rightarrow}} \\
g_j = |\mathbf{C}'_j| - \tau_j^2 u_{\underset{j}{\rightarrow}\underset{j}{\leftarrow}} u_{\underset{j}{\leftarrow}\underset{j}{\leftarrow}} & g'_j = |\mathbf{D}'_j| - \tau_j^2 u_{\underset{j}{\rightarrow}\underset{j}{\rightarrow}} u_{\underset{j}{\leftarrow}\underset{j}{\rightarrow}} \\
h_j = \mathrm{Tr}\,(\mathbf{C}'_j \hat{\mathbf{C}}''_j) & h'_j = \mathrm{Tr}\,(\mathbf{D}'_j \hat{\mathbf{D}}''_j) \\
\mathbf{C}'_j = \mathbf{J}_{\underset{j}{\rightarrow}\underset{j}{\leftarrow}} \mathbf{A}'_j + \mathbf{A}'_j \mathbf{J}_{\underset{j}{\leftarrow}\underset{j}{\leftarrow}} & \mathbf{D}'_j = \mathbf{J}_{\underset{j}{\rightarrow}\underset{j}{\rightarrow}} \mathbf{B}'_j + \mathbf{B}'_j \mathbf{J}_{\underset{j}{\leftarrow}\underset{j}{\rightarrow}} \\
\mathbf{C}''_j = \mathbf{J}_{\underset{j}{\rightarrow}\underset{j}{\leftarrow}} \mathbf{A}''_j + \mathbf{A}''_j \mathbf{J}_{\underset{j}{\leftarrow}\underset{j}{\leftarrow}} & \mathbf{D}''_j = \mathbf{J}_{\underset{j}{\rightarrow}\underset{j}{\rightarrow}} \mathbf{B}''_j + \mathbf{B}''_j \mathbf{J}_{\underset{j}{\leftarrow}\underset{j}{\rightarrow}}
\end{array}
$$

$$
\begin{aligned}
\mathbf{A}'_j &= r_{jj}((\mathbf{J}_{\underset{j}{\rightarrow}j} + \mathbf{J}_{\underset{j}{\leftarrow}j})\mathbf{J}_{\underset{j}{\leftarrow}\underset{j}{\rightarrow}} - \mathbf{J}_{\underset{j}{\rightarrow}\underset{j}{\rightarrow}}(\mathbf{J}_{\underset{j}{\rightarrow}j} + \mathbf{J}_{\underset{j}{\leftarrow}j})) \\
\mathbf{A}''_j &= r_{jj}((\mathbf{J}_{\underset{j}{\rightarrow}j} \mathbf{J}_{\underset{j}{\leftarrow}j} - \mathbf{I})\mathbf{J}_{\underset{j}{\leftarrow}\underset{j}{\rightarrow}} - \mathbf{J}_{\underset{j}{\rightarrow}\underset{j}{\rightarrow}}(\mathbf{J}_{\underset{j}{\rightarrow}j} \mathbf{J}_{\underset{j}{\leftarrow}j} - \mathbf{I})) \\
\mathbf{B}'_j &= r_{jj}((\mathbf{J}_{\underset{j}{\rightarrow}j} + \mathbf{J}_{\underset{j}{\leftarrow}j})\mathbf{J}_{\underset{j}{\leftarrow}\underset{j}{\leftarrow}} - \mathbf{J}_{\underset{j}{\rightarrow}\underset{j}{\leftarrow}}(\mathbf{J}_{\underset{j}{\rightarrow}j} + \mathbf{J}_{\underset{j}{\leftarrow}j})) \\
\mathbf{B}''_j &= r_{jj}((\mathbf{J}_{\underset{j}{\rightarrow}j} \mathbf{J}_{\underset{j}{\leftarrow}j} - \mathbf{I})\mathbf{J}_{\underset{j}{\leftarrow}\underset{j}{\leftarrow}} - \mathbf{J}_{\underset{j}{\rightarrow}\underset{j}{\leftarrow}}(\mathbf{J}_{\underset{j}{\rightarrow}j} \mathbf{J}_{\underset{j}{\leftarrow}j} - \mathbf{I})).
\end{aligned}
$$

There are 3 possible ways of resolving ambiguity in (13). Firstly, ϕ_j can be introduced into parameterization as additional dependent parameters. Secondly, one can use 3 additional boolean parameters to indicate "+" or "−" in ϕ_j. And other way is to invert signs of β_{ij}, α_{ij} and μ_j, if ϕ_j has "+", then change it to "−".

Finally, resulting parameterization has 25 parameters: 3 μ_j, 9 α_{ij}, 9 β_{ij}, 3 u_{jj} and coupling asymmetry l. 21 of them are independent, and there are 4 identities which can be obtained from (11) and (12)

$$
\begin{aligned}
&\mathrm{Tr}\,(\mathbf{W}_{jj}\mathbf{J}_{\underset{j}{\leftarrow}\underset{j}{\leftarrow}}\hat{\mathbf{W}}_{jj}\mathbf{J}_{\underset{j}{\rightarrow}\underset{j}{\rightarrow}} - \mathbf{W}_{jj}\mathbf{J}_{\underset{j}{\leftarrow}\underset{j}{\rightarrow}}\hat{\mathbf{W}}_{jj}\mathbf{J}_{\underset{j}{\rightarrow}\underset{j}{\leftarrow}}) = \\
&\quad = \tau_j(u_{\underset{j}{\rightarrow}\underset{j}{\leftarrow}}u_{\underset{j}{\leftarrow}\underset{j}{\leftarrow}} - u_{\underset{j}{\rightarrow}\underset{j}{\rightarrow}}u_{\underset{j}{\leftarrow}\underset{j}{\rightarrow}}) \\
&4\mathrm{Tr}\,(\mathbf{W}_{11}\mathbf{W}_{22}\mathbf{W}_{33}) = -l^2(1+s) + \\
&\quad +(1-s+2u_{11})(1-s+2u_{22})(1-s+2u_{33})\,,
\end{aligned}
$$

where $s = u_{11}+u_{22}+u_{33}$.

4D CASE

This case can be deduced from 6D case with $\mathbf{M}_{31} = \mathbf{M}_{13} = \mathbf{M}_{32} = \mathbf{M}_{23} = \mathbf{0}$. The only nonzero $\mathbf{W}_{ij}$ are $\mathbf{W}_{32} = -\mathbf{W}_{31} = \mathbf{W}$, and all u_{ij} depend on one parameter u: $u_{11} = u_{22} = 1 - u$, $u_{12} = u_{21} = u$. Then

$$
\mathbf{W}\mathbf{J}_{21} = \mathbf{J}_{11}\mathbf{W}, \quad \mathbf{W}\mathbf{J}_{22} = \mathbf{J}_{12}\mathbf{W}, \quad |\mathbf{W}| = u(1-u). \tag{14}
$$

If $u \neq \{0; 1\}$, then this system can be solved only in the case of $\mathrm{Tr}\,(\mathbf{J}_{11}\mathbf{J}_{12} - \mathbf{J}_{21}\mathbf{J}_{22}) = 0$. So, the following identity can be derived

$$
\beta_{11}\gamma_{12}+\beta_{12}\gamma_{11}-2\alpha_{11}\alpha_{12} = \beta_{21}\gamma_{22}+\beta_{22}\gamma_{21}-2\alpha_{21}\alpha_{22}\,. \tag{15}
$$

Solution of (14) is the following

$$
\mathbf{W} = k\left(\mathbf{J}_{11}\left(\mathbf{J}_{12}-\mathbf{J}_{22}\right) + \left(\mathbf{J}_{12}-\mathbf{J}_{22}\right)\mathbf{J}_{21}\right)\,,
$$

where

$$
k = \pm\sqrt{\frac{u(1-u)}{\left|\mathbf{J}_{11}\left(\mathbf{J}_{12}-\mathbf{J}_{22}\right) + \left(\mathbf{J}_{12}-\mathbf{J}_{22}\right)\mathbf{J}_{21}\right|}}\,.
$$

If of $k < 0$ then its sign should be changed along with simultaneous inversion of signs of $\{\mathbf{J}_{11}, \mathbf{J}_{21}, \mu_1\}$ or $\{\mathbf{J}_{12}, \mathbf{J}_{22}, \mu_2\}$ to resolve ambiguity.

The parameter set for 4D case is also redundant and has 11 items: 2 μ_j, 4 α_{ij}, 4 β_{ij} and u. Only 10 of them are independent because of identity (15).

RELATION TO LEBEDEV—BOGACZ PARAMETERIZATION

One can easily set up a correspondence between notations used in Lebedev—Bogacz parameterization [4] (left) and in the parameterization described above (right)

$$
\begin{array}{ll}
\mu_{1L} = \mu_1 \cdot \mathrm{sgn}\,\beta_{11} & \mu_{2L} = \mu_2 \cdot \mathrm{sgn}\,\beta_{22} \\
\beta_{1x} = |(1-u)\beta_{11}| & \beta_{1y} = |u\beta_{21}| \\
\beta_{2x} = |u\beta_{12}| & \beta_{2y} = |(1-u)\beta_{22}|
\end{array}
$$

$$
\begin{aligned}
\alpha_{1x} &= (1-u)\alpha_{11} \cdot \mathrm{sgn}\,((1-u)\beta_{11}) \\
\alpha_{2x} &= u\alpha_{12} \cdot \mathrm{sgn}\,(u\beta_{12}) \\
\alpha_{1y} &= u\alpha_{21} \cdot \mathrm{sgn}\,(u\beta_{21}) \\
\alpha_{2y} &= (1-u)\alpha_{22} \cdot \mathrm{sgn}\,((1-u)\beta_{22})
\end{aligned}
\quad ,
$$

the "L" index of μ_{1L} and μ_{2L} is introduced to emphasize possible sign change. There are two main differences. Firstly, in [4] α_{ij} and β_{ij} depend on u, and u is dependent parameter, which, in turn, can be expressed through α_{ij} and β_{ij}. In the present paper α_{ij} and β_{ij} are independent on u, but there is identity (15). Secondly, in [4] all β_{ij} are positive, but there are 2 additional boolean parameters. In this paper these ambiguities are resolved by lifting the restriction of $\beta_{ij} > 0$.

SECOND-MOMENTS MATRIX AND EMITTANCES

If $\mathbf{\Sigma}$ is the second-moments matrix, then $\mathbf{\Sigma} = \mathbf{M}\mathbf{\Sigma}\mathbf{M}^T$. Using the following notation

$$
\tilde{\mathbf{\Sigma}}_j = \mathrm{diag}\left(\tilde{\mathbf{\Sigma}}_{1j} \quad \tilde{\mathbf{\Sigma}}_{2\underset{j}{\rightarrow}} \quad \tilde{\mathbf{\Sigma}}_{3\underset{j}{\leftarrow}}\right)\,, \quad \tilde{\mathbf{\Sigma}}_{ij} = -\varepsilon_j \mathbf{J}_{ij}\mathbf{S}\,,
$$

one obtains closed expression for $\mathbf{\Sigma}$

$$
\mathbf{\Sigma} = \overset{\leftrightarrow}{\mathbf{W}}_1\,\tilde{\mathbf{\Sigma}}_1 + \overset{\leftrightarrow}{\mathbf{W}}_2\,\tilde{\mathbf{\Sigma}}_2 + \overset{\leftrightarrow}{\mathbf{W}}_3\,\tilde{\mathbf{\Sigma}}_3\,.
$$

Here ε_j are emittances of normal modes, one can calculate them from beam sizes

$$
\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{pmatrix} = \begin{pmatrix} u_{11}\beta_{11} & u_{12}\beta_{12} & u_{13}\beta_{13} \\ u_{21}\beta_{21} & u_{22}\beta_{22} & u_{23}\beta_{23} \\ u_{31}\beta_{31} & u_{32}\beta_{32} & u_{33}\beta_{33} \end{pmatrix}^{-1} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix}.
$$

ACKNOWLEDGEMENT

The author is grateful to D.N. Shatilov, E.B. Levichev, E.A. Perevedentzev, P.A. Piminov, N.A. Vinokurov (BINP) for review comments and discussions.

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ISBN 978-3-95450-181-6