FORMATION OF A GIVEN DISTRIBUTION OF THE BEAM IN THE PERIODIC CHANNEL

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Abstract

Accelerators and beam transport systems are the most widely used in various fields of fundamental and applied science. This leads to the need for construction of correct models of the corresponding objects and processes. In this paper we consider the problem of forming a beam with a given distribution. Note that similar requirement to the beam occurs in circular accelerators, and in the beam transport systems for various fields of physics, chemistry, biology and so on. Special attention is paid to the development of effective mathematical methods and computer programs to modeling of control systems to ensure the necessary requirements to the beam.

INTRODUCTION

At the moment, we have a long experience in the study of the dynamics of the distribution of particles in the beam in a periodic channel (see, eg, [1], [2], [3]). It is well known that variety of self-consistent density distributions can be built in a uniform focusing channel, see [4], [5]. However, the question of purposeful formation of the desired particle distribution (for example, in the configuration space) in our opinion was not sufficiently developed in theory and practice of accelerator systems, despite the relatively high demand for similar tasks. Requirements for particle beams in modern accelerators, are constantly increasing. First of all, it concerns aspects of modeling optimal systems. Secondly, the inclusion of non-linear control elements for targeted management to produce beams with the specified parameters.

In this direction there are some "basic" approaches. The first type (direct) is based on a careful study of the influence of various factors on the beam dynamics. The second method uses optimization techniques, in which there essentially are solved the inverse problems. Indeed, in this case, we can formalize the requirements for the beam parameters and to implement the search process of optimal solutions for ensuring specified requirements with a given accuracy. It should be noted that inverse problems are a class of ill-posed problems requiring the construction of special methods of solution. However, in both methods there are some common features. In the following we briefly describe the formalism in which the beam itself and the control system of the accelerator are described. This formalism is based on an approach based on the matrix representation for evolution operators [5] on the one hand and different forms of descriptions of the beam as a collective object.

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CONCEPT OF MATRIX FORMALISM

Let us briefly describe the essential features of the mathematical formalism of the matrix formalism and its features that allow realizing for effective computational experiments. Following [5] for the nonlinear ordinary differential equations describing the evolution of the particles in the accelerator we can write

$$\frac{d\mathbf{X}}{ds} = \mathbf{F}^{\mathrm{ext}}(\mathbf{B}^{\mathrm{ext}}(\mathbf{X},s,\mathbf{E}^{\mathrm{ext}}(\mathbf{X},s),\mathbf{X},s)+ \mathbf{F}^{\mathrm{self}}(\langle f(\mathbf{X},s)\rangle_{\mathfrak{M}},\mathbf{X},s), \quad (1)$$

where $\langle f(\mathbf{X},s)\rangle_{\mathfrak{M}}$ means that the distribution function is included in the $\mathbf{F}$ via integral and the integration is over volume occupied by particles $\mathfrak{M} = \mathfrak{M}(s)$, where s is the length measured along the reference orbit. According to [5] one can introduce the evolution operator $\mathcal{M}(t\,|\,t_0;\mathbf{F})\colon \mathbf{X}_0 \to \mathbf{X}(t)$ in by equation (1). Given the property of the evolution operator one can write

$$\mathbf{F}^{\mathrm{self}}(\langle f(\mathbf{X},t)\rangle_{\mathfrak{M}},\mathbf{X},t) = \mathbf{F}^{\mathrm{self}}(\langle f_0(\mathcal{M}^{-1}(t\,|\,t_0,\mathbf{F})\circ\mathbf{X}_0)\rangle_{\mathfrak{M}_0},\mathbf{X},t). \quad (2)$$

In accordance with the equations (1) and (2) one can write the following integral operator equation

$$\mathcal{M}\left(t\,|\,t_0;\mathcal{V}^{\mathrm{ext}}+\mathcal{V}^{\mathrm{self}}\right) = \mathcal{I}d + \int\limits_{t_0}^{t}\left(\mathcal{V}^{\mathrm{ext}}(\tau)+ \mathcal{V}^{\mathrm{self}}(\tau)\right)\circ\mathcal{M}\left(\tau\,|\,t_0;\mathcal{V}^{\mathrm{ext}}(\tau)+\mathcal{V}^{\mathrm{self}}(\tau)\right)d\tau. \quad (3)$$

We note that equation (3) is an integral equation of Volterra–Urysohn of type II (see, eg, [6]), as in fact the control system (the transportation system) – the control object (beam) is covered by the feedback. It proves that the sequence $\mathcal{M}^k$ converges (in some sense) to some element $\mathcal{M}^{\mathrm{fin}}$, and one can install the identity $\mathcal{M}^{\mathrm{fin}} = \mathcal{A}\circ\mathcal{M}^{\mathrm{fin}}$. We should note that similar approach not only allows us to use different forms describe the beam dynamics, taking into account the impact of its own charge but and allows us to build different methods of description of the beam dynamics, taking into account the impact of their own charge. The similar approach allows us to build different forms describe the beam dynamics, taking into account the impact

ISBN 978-3-95450-181-6

of their own charge. In particular, for a given initial distribution function we can introduce different presentation of the generalized matrix of envelopes

$$\mathfrak{S}_0^{ik} = \int\limits_{\mathfrak{M}_0} f_0(\mathbf{X})\mathbf{X}^{[i]}\left(\mathbf{X}^{[k]}\right)^{\mathrm{T}} d\mathbf{X},$$

where the $\mathfrak{M}_0$ is an initial set in the phase space.

As an example, consider the case of the elliptical beam

$$\mathbf{G}_0(\mathbf{X},\varepsilon) = \mathbf{X}^{\mathrm{T}}\,\mathbb{A}_0\,\mathbf{X} - \varepsilon, \quad 0 < \varepsilon \le 1,$$

where $\mathbb{A}_0$ is the matric defining the initial beam distribution. We shall seek the solution of equations of motion taking into account the space charge in the following form $\mathbf{X}(t) = \sum\limits_{k=1}^{N} \mathbb{M}^{1k}(t|t_0)\mathbf{X}_0^{[k]}$, and correspondingly $\mathbf{X}_0 = \sum\limits_{k=1}^{N} \mathbb{T}^{1k}(t|t_0)\mathbf{X}^{[k]}(t)$, where $\mathbb{M}^{1k}$ and $\mathbb{T}^{1k}$ are polynomials of k-th order.

In this case, the transformation matrix of the envelopes can be represented as follows

$$\mathbf{G}(\mathbf{X},t) = \left(\sum_{k=1}^{\infty}\mathbb{T}^{1k}\mathbf{X}^{[k]}\right)^{\mathrm{T}}\mathbb{A}_0\left(\sum_{j=1}^{\infty}\mathbb{T}^{1j}\mathbf{X}^{[j]}\right) - \varepsilon =$$
$$\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\left(\mathbf{X}^{[k]}\right)^{\mathrm{T}}\left(\left(\mathbb{T}^{1k}\right)^{\mathrm{T}}\mathbb{A}_0\mathbb{T}^{1j}\right)\mathbf{X}^{[j]} - \varepsilon =$$
$$\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\left(\mathbf{X}^{[k]}\right)^{\mathrm{T}}\mathbb{A}^{kj}\mathbf{X}^{[j]} - \varepsilon = \left(\mathbf{X}^{\infty}\right)^{\mathrm{T}}\mathbb{A}^{\infty}\mathbf{X}^{\infty} - \varepsilon, \quad (4)$$

$$\mathbb{A}^{\infty} = \left(\mathbb{A}^{kj}\right)_{k,j\ge 1}, \quad \mathbb{A}^{kj} = \left(\mathbb{T}^{1k}\right)^{\mathrm{T}}\mathbb{A}_0\mathbb{T}^{1j},$$

where $\mathbb{A}^{\infty}$ is a matrix definding the current distribution of particles in beam considering the nonlinear effects. The equality (4) one can write as the following equation

$$\frac{d\mathfrak{S}^{ik}}{dt} = \sum_{l=i}^{\infty}\mathbb{P}^{il}\mathfrak{S}^{lk} + \sum_{l=k}^{\infty}\mathfrak{S}^{lk}\left(\mathbb{P}^{kl}\right)^*,$$

or by combining, for $\mathfrak{S}^{\infty}$:

$$\frac{d\mathfrak{S}^{\infty}(t)}{dt} = \mathbb{P}^{\infty}(t)\mathfrak{S}^{\infty}(t) + \mathfrak{S}^{\infty}(t)\left(\mathbb{P}^{\infty}(t)\right)^*,$$

where, as before, $\mathbb{P}^{\infty}(t)$ is an upper triangular matrix, $\mathbb{P}^{ik}$ ($\mathbb{P}^{ik} \equiv 0\ \forall\ i > k$) are the block matrices.

The initial matrix $\mathfrak{S}_0^{ik}$ or $\mathfrak{S}_0^{\infty}$ can be calculated according to equations

$$\mathfrak{S}_0^{ik} = \int\limits_{\mathfrak{M}_0} f(\mathbf{X}_0)\mathbf{X}_0^{[i]}\left(\mathbf{X}_0^{[k]}\right)^* d\mathbf{X}_0,$$

$$\mathfrak{S}_0^{\infty} = \int\limits_{\mathfrak{M}_0} f(\mathbf{X}_0)\mathbf{X}_0^{\infty}\left(\mathbf{X}_0^{\infty}\right)^* d\mathbf{X}_0.$$

After this we implement the method of successive approximations using the method of virtual changing of beam parameters $\mathfrak{S}_1^{ik} = \alpha\mathfrak{S}_0^{ik} + (1-\alpha)\,\mathfrak{S}_0^{ik}$, $0 < \alpha < 1$. This process is ended when there the following inequality

$$2\frac{\|\mathfrak{S}_1^{ik} - \mathfrak{S}_0^{ik}\|_c}{\|\mathfrak{S}_1^{ik} + \mathfrak{S}_0^{ik}\|_c} < \varepsilon^{ik}$$

where for any enough small $\varepsilon^{ik} \ll 1$. We should note that the order of nonlinearities is defined by our knowledges about the transport system (see the next section).

We also note that one can use the some another presentatiom for beam description (see, for example, [5]), in particular using a particle distribution function $f(\mathbf{X},t)$.

BEAM FORMING WITH ALMOST UNIFORM DISTRIBUTION

Let be the next function describe some particles distribution in phase space

$$f_0(\mathbf{X}) = Q_{2m}(\mathbf{X})e^{-P_{2n}(\mathbf{X})}, \quad (5)$$

where $Q_{2m}(\mathbf{X})$, $P_{2n}(\mathbf{X})$ are polynomials of the $2m$-th and $2n$-th order correspondingly. Coefficients of these polynomials are determined from experimental data fully or partly. In the last case the free coefficients can be determined from other information or used as control parameters. At the base of our approach we use the methods which briefly described in the previous sections.

According to [5] one can write

$$f(\mathbf{X},t) = P_{N_1}\left(\sum_{k=1}^{N}\sum_{j=1}^{N}\left(\mathbf{X}^{[k]}\right)^{\mathrm{T}}\left(\mathbb{T}^{1k}\right)^{\mathrm{T}}\mathbb{A}_0\mathbb{T}^{1j}\mathbf{X}^{[j]}\right)\times$$
$$\exp\left(-\frac{Q_{N_2}\left(\sum\limits_{j=1}^{N}\left(\mathbf{X}^{[k]}\right)^{\mathrm{T}}\left(\mathbb{T}^{1k}\right)^{\mathrm{T}}\mathbb{A}_0\mathbb{T}^{1j}\mathbf{X}^{[j]}\right)}{2\sigma^2}\right). \quad (6)$$

As an example, let us consider the formation of the current distribution of the beam in the nonlinear channel (to the third order of nonlinearities). For testing (besides of the usual computer codes) we also use systems for symbolic computation (packages Mathematica and Maple) for the analysis of possible functions of beam distributions. At the first stage, we search the matrix elements included in the matrices $\mathbb{M}^{1k}$ for $1 \le k \le n$ for some n (in our example we restricted by $n = 3$). In general the researcher must solve the inverse problem of determining the values of the control parameters for the channel transport in view of the desired output beam distribution in coordinate space. This task can be formulated as a problem with the feedback and demands quite time-consuming necessary for computing that leads us to using of modern computer systems. As an example, we consider the formation of the current distribution of the beam in the nonlinear channel (up to third

ISBN 978-3-95450-181-6

order of nonlinearity). Using packages Mathematica and Maple help us to provide visual representation of the calculations for the analysis of possible beam distributions. At the first stage, the search matrix elements for the matrices $\mathbb{M}^{1k}$ for $1 \leq k \leq n$ for some n (in our example we restricted by third order). The investigator should solving the inverse problem for determining the values of the control parameters for the channel transport in view of the desired output beam distribution in coordinate space. As this problem with the feedback, then it leads us to enough time-consuming computing. That is why it is necessary usage modern computer systems with parallel and distributed operations.

CONCLUSION

In the conclusion we should note that computational accelerator physics has significantly changed and is broadened over the last years. These changes are first of all due to the advent new methods of high performance computing (parallel and distributed) and in connection with the capabilities of modern packages for symbolic computations. In particular many new mathematical methods appeared namely largely due to the packages for symbolic computations. These new models and instruments can provide not only overall accuracy of mathematical and computer modelling, but better understanding of the physical processes in accelerators. Finally, the use of modern computational frameworks provide an opportunity to developers to concentrate first of all on the mathematics and physics. In the article we give some pictures (see Fig. 1, Fig. 2, Fig. 3) that not only demonstrate possibilities of above described approaches, but and demonstrate the transfer from Gauss distribution into a distribution with a nearly uniform distribution (in the configuration space). The necessary computations were realized up to third order of nonlinearity. We also note that we considered the beam parameters and the structure of cyclic accelerator with parameters are enough close to the structure of the Nuclotron in JINR (Russia).

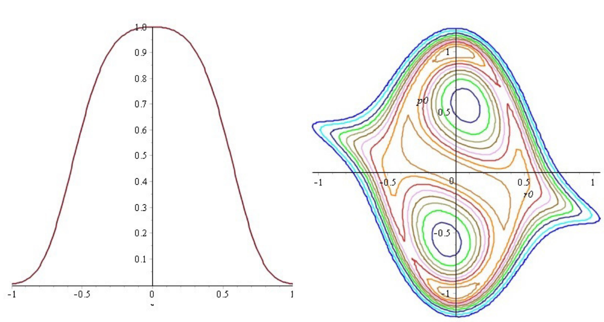

Figure 1: On the picture there are presented: on the left – an almost Gauss distribution of the beam in the x-section (as an example); on the right – the example of beam transformation after tens of thousands of revolutions in the $x - p_x$ section (view from "above").

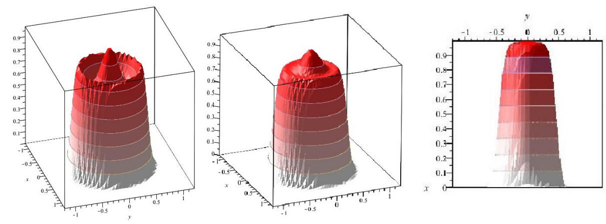

Figure 2: On the picture the successive changes of distribution of beam particles are presented (in the $x - y$ plane). The left two pictures present the sequential (in time) changing of distribution for particles of beam in x- and y-planes. The almost uniform distribution (the goal of the transformations) is demonstrated on the third picture.

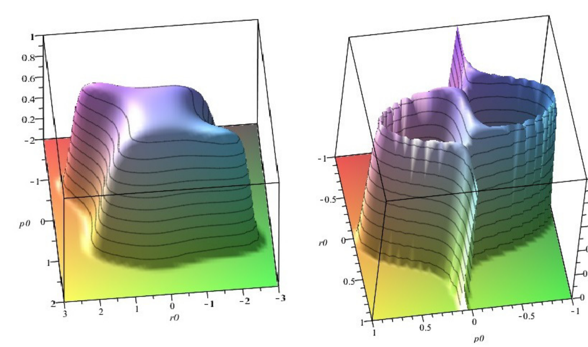

Figure 3: On the picture there are presented examples of distribution of particles in the $x - p_x$ plane. On the left picture – an intermediate stage of the fillamentation process, on the right picture – some next stage of the fillamentation process (after thousands of turns of the beam evolution).

REFERENCES

[1] N.Pichoff, G.Haouat, P.Y.Beauvais, J.M.Lagniel, R.Ferdinand, S.Nath, S.Joly Transverse-profile equilibrium in a space-charge-dominated beam 1st Asian Particle Accelerator Conference, Tsukuba, Japan, 23 - 27 Mar 1998, pp.617

[2] R.A.Kishek, S.Bernal, P.G.O Shea, M.Reiser, I. Haber Transverse space-charge modes in non-equilibrium beam Nuclear Instruments and Methods in Physics Research A 464 P. 484-492, 2001.

[3] J.Struckmeier, I.Hofmann Generation of emittance conserving non-kv distributions in periodic focusing channels. http://accelconf.web.cern.ch/AccelConf/e98/PAPERS/THP43G.PDF

[4] M.Reiser Theory and Design of Charged Particle Beams. Wiley-VCH Verlag GmbH & Co. KGaA 2008

[5] S.N. Andrianov, Dynamical Modeling of Particle Beam Control Systems. SPbSU. SPb. 2004 (in Russian).

[6] Urysohn, P. S. On a type of nonlinear integral equation. Mat. Sb., 31. P. 236–355, 1924.

ISBN 978-3-95450-181-6