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PHASE SPACE GYMNASTICS

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Abstract

Phase space gymnastics is a highly evolved accelerator physics technique based on the finest properties of the phase space. As modern accelerators become increasingly demanding, these techniques are finding a sharp increase in their applications. Here we intend only to introduce this topic to bring attention to the direction it points to.

INTRODUCTION

As accelerator technology advances, the requirements on accelerator beam quality become increasingly demanding. Facing these new demands, the topic of phase space gymnastics is becoming a new focus of accelerator physics research. In a phase space gymnastics, the beam's phase space distribution is manipulated and precision-tailored to meet the required beam qualities. On the other hand, all realization of such gymnastics will have to obey accelerator physics principles as well as technological limitations. Recent examples of phase space gymnastics include

1. Adapters
2. Emittance exchanges
3. Phase space exchanges
4. Emittance partitioning
5. Seeded free electron lasers
6. Steady-state microbunched storage rings

Each one of these applications involves half a dozen to a dozen inventions to special cases. It can only be expected that many more applications are yet to be found. This research filed is very rich and active. In this report, however, we aim only to illustrate the subject and we will only breifly address the case of adapters (item 1 above) and give some of their example applications.

Just like the physical gymnastics, e.g. in the Olympic games, the skills needed in phase space gymnastics are highly technical and precise, while the resulting performance exquisite and beautiful. A comparison of these two gymnastics skills is shown in Fig. 1. Earlier phase space gymnastics have been mostly applied to the 2D longitudinal phase space, and took the form of RF manipulations in beam injection, extraction, and phase space displacement acceleration [1]. The recent advances, led by the seminal papers by Derbenev [2], begin to incorporate the transverse dimensions and become much more sophisticated, yielding a new wealth of additional applications mentioned above.

It should be mentioned here that phase space gymnastics permit precision manipulations because phase space is conserved to its finest details. Liouville theorem (more accurately, the condition of symplecticity) is the root cause of this possibility of phase space technology. The very concept of phase space (a bold extension and abstraction of the 3D real space), and its intricate physical and mathematical properties (that pave the foundation of these phase space techniques), however, are not the subject of this report.

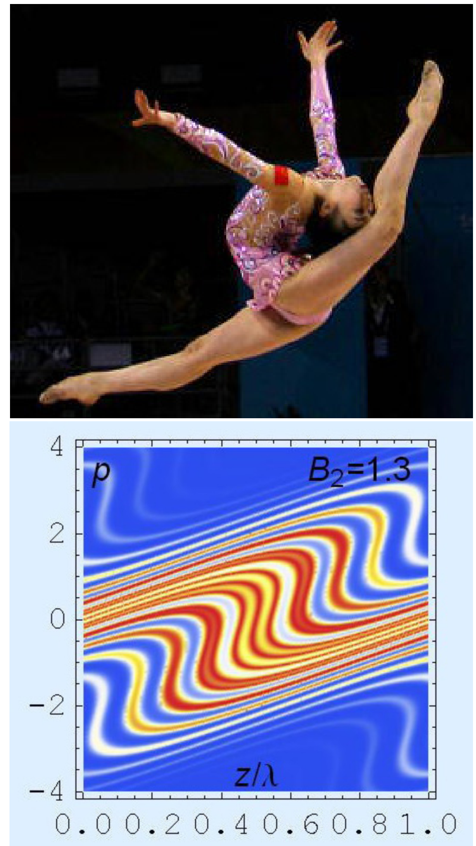

Figure 1: A comparison of phase space gymnastics and physical gymnastics.

ADAPTERS

The idea of adapters was first introduced by Derbenev [2] and later rapidly extended by him and many others [3]-[14]. A few adapters of a different variety are shown in Fig. 2. Derbenev first envisioned applying it to a storage ring collider to form round beams at the collision point to mitigate the effect of the encountered beam-beam nonlinear resonances. This adapter idea has also been adapted for electron cooling [3,7]. Furthermore, the production of a very flat beam from a round photocathode immersed in a solenoid followed by a round-to-flat adapter has been experimentally demonstrated [10–12].

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Figure 2: A few adapters of a different variety.

During this time, as mentiond earlier, several other kinds of adapters have been invented, including emittance exchange adapters and phase space exchange adapters and nonsymplectic applications for emittance partitioning have also been developed, but we do not cover those applications below.

Consider the 4D canonical phase space $X_{\text{can}} = (x, p_x, y, p_y)$. We have two representations to describe particle motion in this phase space:

1. For an uncoupled case, use the Courant-Snyder basis of planar modes (x and y modes) in a matrix form:

$$X_{\text{can}} = Va \tag{1}$$

where

$$V = \begin{bmatrix} \sqrt{\beta_x}\cos\phi_x & \sqrt{\beta_x}\sin\phi_x & 0 & 0 \\ \frac{-\alpha_x\cos\phi_x - \sin\phi_x}{\sqrt{\beta_x}} & \frac{-\alpha_x\sin\phi_x + \cos\phi_x}{\sqrt{\beta_x}} & 0 & 0 \\ 0 & 0 & \sqrt{\beta_y}\cos\phi_y & \sqrt{\beta_y}\sin\phi_y \\ 0 & 0 & \frac{-\alpha_y\cos\phi_y - \sin\phi_y}{\sqrt{\beta_y}} & \frac{-\alpha_y\sin\phi_y + \cos\phi_y}{\sqrt{\beta_y}} \end{bmatrix} \tag{2}$$

and

$$a = \begin{bmatrix} \sqrt{2\epsilon_x}\sin\chi_x \\ \sqrt{2\epsilon_x}\cos\chi_x \\ \sqrt{2\epsilon_y}\sin\chi_y \\ \sqrt{2\epsilon_y}\cos\chi_y \end{bmatrix} \tag{3}$$

These equations describe the motion of a particle in a planar beamline whose x and y emittances are ϵ_x, ϵ_y and initial betatron phases of χ_x and χ_y. Lattice functions $\alpha_{x,y}, \beta_{x,y}, \phi_{x,y}$ are the familiar betatron parameters in this representation. Indeed by direct multiplication, we have

$$x_{\text{can}} = Va = \begin{bmatrix} \sqrt{2\beta_x\epsilon_x}\sin(\phi_x + \chi_x) \\ \sqrt{\frac{2\epsilon_x}{\beta_x}}[\cos(\phi_x + \chi_x) - \alpha_x\sin(\phi_x + \chi_x)] \\ \sqrt{2\beta_y\epsilon_y}\sin(\phi_y + \chi_y) \\ \sqrt{\frac{2\epsilon_y}{\beta_y}}[\cos(\phi_y + \chi_y) - \alpha_y\sin(\phi_y + \chi_y)] \end{bmatrix} \tag{4}$$

Equation (4) is a very familar result. What may be less familiar, and it might come as a surprise, is the fact that this familiar result is actually factorizable according to Eq. (1).

2. For a fully coupled beam with rotational symmetry (e.g. in a solenoidal field), one can describe particle motion using the basis of circular modes (left-handed and right handed modes) [8]:

$$X_{\text{can}} = Ub \tag{5}$$

where

$$U = \frac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{\beta}\cos\phi_+ & \sqrt{\beta}\sin\phi_+ & -\sqrt{\beta}\cos\phi_- & -\sqrt{\beta}\sin\phi_- \\ \frac{-\sin\phi_+ - \alpha\cos\phi_+}{\sqrt{\beta}} & \frac{\cos\phi_+ - \alpha\sin\phi_+}{\sqrt{\beta}} & \frac{\sin\phi_- + \alpha\cos\phi_-}{\sqrt{\beta}} & \frac{-\cos\phi_- + \alpha\sin\phi_-}{\sqrt{\beta}} \\ -\sqrt{\beta}\sin\phi_+ & \sqrt{\beta}\cos\phi_+ & -\sqrt{\beta}\sin\phi_- & \sqrt{\beta}\cos\phi_- \\ \frac{-\cos\phi_+ + \alpha\sin\phi_+}{\sqrt{\beta}} & \frac{-\sin\phi_+ - \alpha\cos\phi_+}{\sqrt{\beta}} & \frac{-\cos\phi_- + \alpha\sin\phi_-}{\sqrt{\beta}} & \frac{-\sin\phi_- - \alpha\cos\phi_-}{\sqrt{\beta}} \end{bmatrix} \tag{6}$$

and

$$b = \begin{bmatrix} \sqrt{2\epsilon_+}\sin\chi_+ \\ \sqrt{2\epsilon_+}\cos\chi_+ \\ \sqrt{2\epsilon_-}\sin\chi_- \\ \sqrt{2\epsilon_-}\cos\chi_- \end{bmatrix} \tag{7}$$

for a particle with left-handed and right-handed emittances ϵ_+ and ϵ_- and initial betatron phases χ_+ and χ_-. Lattice parameters are $\alpha, \beta, \phi_+, \phi_-$, i.e., there is only one β-function, one α-function, but two (left-handed and right-handed) phases. Direct multiplication gives

$$X_{\text{can}} = Ub = \begin{bmatrix} \sqrt{\beta}(\sqrt{\epsilon_+}\sin\Delta_+ - \sqrt{\epsilon_-}\sin\Delta_-) \\ \frac{1}{\sqrt{\beta}}(\sqrt{\epsilon_+}\cos\Delta_+ - \sqrt{\epsilon_-}\cos\Delta_- - \alpha\sqrt{\epsilon_+}\sin\Delta_+ + \alpha\sqrt{\epsilon_-}\sin\Delta_-) \\ \sqrt{\beta}(\sqrt{\epsilon_+}\cos\Delta_+ + \sqrt{\epsilon_-}\cos\Delta_-) \\ -\frac{1}{\sqrt{\beta}}(\sqrt{\epsilon_+}\sin\Delta_+ + \sqrt{\epsilon_-}\sin\Delta_- + \alpha\sqrt{\epsilon_+}\cos\Delta_+ + \alpha\sqrt{\epsilon_-}\cos\Delta_-) \end{bmatrix}$$

where $\Delta_+ = \phi_+ + \chi_+, \Delta_- = \phi_- + \chi_-$.

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Once we have the planar basis V and the circular basis U — both are symplectic — we are now in a position to consider "adapters". So far we considered transverse phase space only. Adapters can also be applied to transverse-longitudinal coupled systems.

Flat-to-flat Adapters

Flat-to-flat adapter from s_1 to s_2 is well known. The job is to design a lattice that provides the map from the basis $V(s_1)$ to the basis $V(s_2)$, i.e. the optics matching from one set of lattice parameters to another. A moment's reflection shows that the needed matching map is given by $V(s_2)V(s_1)^{-1}$, and a simple calculation gives

$$V(s_2)V(s_1)^{-1} = \begin{bmatrix} \sqrt{\frac{\beta_{x2}}{\beta_{x1}}}(\cos\mu_x + \alpha_{x1}\sin\mu_x) & \sqrt{\beta_{x1}\beta_{x2}}\sin\mu_x & 0 & 0 \\ \frac{(\alpha_{x1}-\alpha_{x2})\cos\mu_x-(1+\alpha_{x1}\alpha_{x2})\sin\mu_x}{\sqrt{\beta_{x1}\beta_{x2}}} & \sqrt{\frac{\beta_{x1}}{\beta_{x2}}}(\cos\mu_x - \alpha_{x2}\sin\mu_x) & 0 & 0 \\ 0 & 0 & \sqrt{\frac{\beta_{y2}}{\beta_{y1}}}(\cos\mu_y + \alpha_{y1}\sin\mu_y) & \sqrt{\beta_{y1}\beta_{y2}}\sin\mu_y \\ 0 & 0 & \frac{(\alpha_{y1}-\alpha_{y2})\cos\mu_y-(1+\alpha_{y1}\alpha_{y2})\sin\mu_y}{\sqrt{\beta_{y1}\beta_{y2}}} & \sqrt{\frac{\beta_{y1}}{\beta_{y2}}}(\cos\mu_y - \alpha_{y2}\sin\mu_y) \end{bmatrix} \quad (8)$$

Equation (8) of course is a well known result; $\mu_x = \phi_{x2} - \phi_{x1}$, $\mu_y = \phi_{y2} - \phi_{y1}$ are the betatron phase advances from s_1 to s_2. A particle with initial condition (3) is now brought from position s_1 to position s_2.

Let us make a few side comments here concerning Eqs. (1) and (8):

- Equation (1) is not to be confused with a similar-looking expression $X_{\rm out} = MX_{\rm in}$, which relates the final coordinates $X_{\rm out}$ to the initial coordinates $X_{\rm in}$ through a beamline element with map M and is the job of Eq. (8). This should become apparent when one observes that the final product of the multiplication of Va yields Eq. (4).
- Equation (1) factorizes the particle's motion into a product of a factor V that depends only on the accelerator optics, and a factor a that depends only on the particle's initial conditions. It is important to note that V has nothing to do with the particle while a has nothing to do with the accelerator.
- Now Eq. (8) goes further in factorization. It says that the map that brings the accelerator optics from s_1 to s_2 can be factorized into a factor $V(s_1)^{-1}$ that depends only on the optical properties at s_1 and a factor $V(s_2)$ that depends only on the optical properties at s_2.

The elegance of this formalism should be very apparent.

Round-to-round Adapters

Round-to-round adapter from s_1 to s_2, i.e., from one set of circular lattice parameters to another, is given by the map $U(s_2)U(s_1)^{-1}$. Although the algebra is somewhat involved, it can be shown that the result can be written as

$$U(s_2)U(s_1)^{-1} = R^{-1}(\theta)T \quad (9)$$

where $R(\theta)$ is a rotation matrix with rotation angle θ,

$$R(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & \cos\theta & 0 & \sin\theta \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & -\sin\theta & 0 & \cos\theta \end{bmatrix} \quad (10)$$

and

$$T = \begin{bmatrix} \sqrt{\frac{\beta_2}{\beta_1}}(\cos\mu + \alpha_1\sin\mu) & \sqrt{\beta_1\beta_2}\sin\mu & 0 & 0 \\ \frac{(\alpha_1-\alpha_2)\cos\mu-(1+\alpha_1\alpha_2)\sin\mu}{\sqrt{\beta_1\beta_2}} & \sqrt{\frac{\beta_1}{\beta_2}}(\cos\mu - \alpha_2\sin\mu) & 0 & 0 \\ 0 & 0 & \sqrt{\frac{\beta_2}{\beta_1}}(\cos\mu + \alpha_1\sin\mu) & \sqrt{\beta_1\beta_2}\sin\mu \\ 0 & 0 & \frac{(\alpha_1-\alpha_2)\cos\mu-(1+\alpha_1\alpha_2)\sin\mu}{\sqrt{\beta_1\beta_2}} & \sqrt{\frac{\beta_1}{\beta_2}}(\cos\mu - \alpha_2\sin\mu) \end{bmatrix} \quad (11)$$

The left-handed and right-handed betatron phases at s_2 are then given by $\phi_{+2} = \phi_{+1} + \mu - \theta$ and $\phi_{-2} = \phi_{-1} + \mu + \theta$.

There are two ways to realize this desired map (9):

- a quadrupole channel that provides the map (11), followed by rotating the entire subsequent beamline (not including the quadrupole channel) by θ.
- A uniform solenoid with strength k_s and length L (including its two ends) will produce this map with $\theta = -k_sL/2, \mu = k_sL/2, \beta_1 = \beta_2 = 2/k_s, \alpha_1 = \alpha_2 = 0,$

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where the solenoid strength is specified by $k_s = \frac{B_s}{(B\rho)_0}$ with B_s the solenoid magnetic field and $(B\rho)_0$ the magnetic rigidity of the electron beam.

Round-to-flat Adapters

A round-to-flat adapter is given by the map $V(s_2)U(s_1)^{-1}$, which can be shown to have a general form of a round-to-round adapter, followed by a specific round-to-flat insertion with map $(VU^{-1})_0$, followed by a flat-to-flat adapter.

These three adapters have the following parameters:

- The first round-to-round transformation is from $(\alpha, \beta, \phi_+, \phi_-)$ to $(\alpha = 0, \beta, \phi_+ = \phi_y + \mu + \pi/4, \phi_- = \phi_y + \mu - \pi/4)$.
- The second round-to-flat transformation $(VU^{-1})_0$ is from $(\alpha = 0, \beta, \phi_+ = \phi_y + \mu + \pi/4, \phi_- = \phi_y + \mu - \pi/4)$ to $(\alpha_x = \alpha_y = 0, \beta_x = \beta_y, \phi_x = \phi_y)$.
- The last flat-to-flat transformation from $(\alpha_x = \alpha_y = 0, \beta_x = \beta_y, \phi_x = \phi_y)$ to $(\alpha_x, \beta_x, \phi_x, \alpha_y, \beta_y, \phi_y)$.

Combining all steps, we then have finally an adapter from $(\alpha, \beta, \phi_+, \phi_-)$ to $(\alpha_x, \beta_x, \phi_x, \alpha_y, \beta_y, \phi_y)$. Each step, although stated in language of mathematics, is directly translatable to conventional lattice designs.

The special round-to-flat map $(VU^{-1})_0$ has a simple form [3, 9, 10]

$$(VU^{-1})_0 = R\left(-\frac{\pi}{4}\right)\begin{bmatrix} -\sqrt{\frac{\beta_y}{\beta}}\sin\mu & -\sqrt{\beta\beta_y}\cos\mu & 0 & 0 \\ \frac{\cos\mu}{\sqrt{\beta\beta_y}} & -\sqrt{\frac{\beta}{\beta_y}}\sin\mu & 0 & 0 \\ 0 & 0 & \sqrt{\frac{\beta_y}{\beta}}\cos\mu & -\sqrt{\beta\beta_y}\sin\mu \\ 0 & 0 & \frac{\sin\mu}{\sqrt{\beta\beta_y}} & \sqrt{\frac{\beta}{\beta_y}}\cos\mu \end{bmatrix} R\left(\frac{\pi}{4}\right) \qquad (12)$$

It is easy to see that $(VU^{-1})_0$ represents a regular quadrupole channel (miminum of three quadrupoles in general) rotated 45°. The 45° rotation renders the quadrupoles skew quadrupoles. Design of the adapter therefore reduces to a regular lattice matching problem.

Inserting a round-to-flat adapter brings a beam from a round optics to a flat optics. A round beam with left-handed and right-handed emittances of (ϵ_+, ϵ_-) is transformed to a planar beam with x- and y-emittances given by $(\epsilon_x = \epsilon_+, \epsilon_y = \epsilon_-)$.

Flat-to-round Adapter

Reversing the round-to-flat adapter, a flat beam with x- and y-emittances of (ϵ_x, ϵ_y) is transformed to a round beam with left-handed and right-handed-emittances $(\epsilon_+ = \epsilon_x, \epsilon_- = \epsilon_y)$. This adapter can also be achieved by an insertion with three skew quadrupoles.

Applications of Flat-to-round and Round-to-flat Adapters

As mentioned, the idea of adapters was first suggested by Derbenev 1993 to control the beam-beam effect in storage ring colliders. But it has subsequently been much extended for other applications.

Storage Ring Colliders In this collider application [2], a planar flat beam in regular arc cells is transformed by a flat-to-round adapter to become a round beam at the collision region. The collision region is immersed in a solenoidal field. After the collision region, the beam is brought back to the regular arc by a round-to-flat adapter. With a round beam at the collision point, this possibly reduces the beam-beam effect due to much reduced number of nonlinear resonances.

Linear Colliders In this application [5], a round beam is produced at the cathode immersed in a solenoidal field. After exiting the solenoid, a round-to-flat adapter transforms the beam into a flat planar configuration, which is what is needed for linear collider applications. The use of adapter here avoids the need of a damping ring to provide flat beams.

Electron Cooling Applying a flat-to-round adapter to a very flat beam ($\epsilon_x \gg \epsilon_y$), a round beam can be produced with $\epsilon_+ \gg \epsilon_-$. Immersing the beam in a matched solenoid with appropriate magnetic field, particles in the beam will move in the solenoid with very small angular divergence, i.e., the beam becomes extremely laminar with all particles moving almost straight ahead along the solenoidal field with zero Larmor radius and as a result almost zero transverse temperature. This is an ideal beam for performing electron cooling [3, 7].

Diffraction Limited Synchrotron Radiation While the application to electron cooling is most likely performed in a linac environment, the same configuration can also be installed in a synchrotron radiation storage ring. By an insertion with the configuration (flat-to-round adapter + solenoid + round-to-flat adapter), a conventional 3rd generation synchrotron radiation storage ring can in principle reach diffraction limit for X-rays [6, 13].

Perhaps one can illustrate the point as follows. To reach diffraction limit for X-rays, much efforts have been dedicated to the design of "ultimate storage rings" aiming for exceedingly small ϵ_x. The beam also has an even much smaller ϵ_y. To reach diffraction limit, we operate the beam fully coupled, so that the coupled beam has both its horizontal and vertical emittances given by the arithmatic mean of ϵ_x and ϵ_y, i.e. they are both equal to $(\epsilon_x + \epsilon_y)/2$. In contrast, the round beam adapter scheme yields a beam with its horizontal and vertical emittances equal to the geometric mean of ϵ_x and ϵ_y, i.e. they are both given by $\sqrt{\epsilon_x\epsilon_y}$. Since $\epsilon_y \ll \epsilon_x$, it is clear that the adapter round beam scheme has a great advantage in reaching small emittances. If, for example, $\epsilon_y = 10^{-3}\,\epsilon_x$, then one potentially gains a factor of 15 reduction in the beam emittances.

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In principle, therefore, the adapter round beam scheme could be applied to conventional 3rd generation storage ring facilities to provide coherent soft X-rays. If so, one can save the effort of developing ultimate rings. In practice, however, the catch is that there is missing a way to match the electron optics to that of the laser optics [15]. As it stands, the requires solenoid field turns out too strong and the required solenoid length is too long.

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