ISBN: 978-3-95450-199-1 doi:10.18429/JACoW-SAP2017-TUPH09

SPONTANEOUS RADIATION OF HIGH-ORDER MAGNETIC FIELD UNDULATOR

Tao Wei †, Peng Li, China Academy of Engineering Physics, Mianyang, China
Joachim Pflueger, Yuhui Li, European XFEL Gmbh, Schenefeld, Germany

Abstract

Based on the purpose of concision, nearly all the undulator radiation formulas have made an assumption that the guiding magnetic field is a sinusoid wave. The assumption is consistent with the truth if the ratio of undulator gap to period length is large enough. However, high-order magnetic field exists widely in most undulators, especially those with long period length and short gap. This paper will derive the radiation output equations of high-order magnetic field undulator, what's more, the formulas are validated through numerical simulation with code SPECTRA.

INTRODUCTION

The undulators have been widely used as insertion devices in synchrotron sources and free electron laser (FEL) to generate magnetic field which is periodic along the electron beam direction. The simplest case is the planar undulator which presents a sinusoid field perpendicular to the electron beam path. It is the most popular undulator model, the characteristic of its radiation have been discussed in many references [1, 2].

Taking into account a practical undulator, high-order magnetic field exists more or less. This paper will derive the far-field radiation of high-order planar undulator and discuss the influence of high-order magnetic field.

SPONTANEOUS RADIATION EQUATION

The magnetic field of planar undulator should be periodic along the beam direction, in addition, the integral of the magnetic field over a single period length vanished [3]. Without loss of generality, the planar undulator which contains high-order magnetic field can be described as

$$\vec{B} = \sum_{m=1}^{+\infty} B_m \sin(mk_u z - \delta\phi_m)\,\overrightarrow{e_y}\,. \tag{1}$$

In which $k_u = 2\pi/\lambda_u$, λ_u is the period length of undulator. To simplify the expression, the phase of fundamental magnetic field was chosen zero ($\delta\phi_1 = 0$).

The electron motion equation in the undulator can be described as

$$\gamma m_e \dot{\vec{v}} = -e\vec{v} \times \vec{B}\,. \tag{2}$$

This results in two coupled equations for the undulator with field distribution (1),

$$\ddot{x} = \frac{e}{\gamma m_e} B_y\,\dot{z}\,, \quad \ddot{z} = -\frac{e}{\gamma m_e} B_y\,\dot{x}\,. \tag{3}$$

In which m_e and e are the mass and charge of electron, γ is the Lorentz factor, z-axis is the direction of electron beam moving forward. The formula (3) can be solved iteratively. To obtain the first-order motion solution we assume v_z keep constant ($v_z = \dot{z} \approx \overline{\beta_s}c$), in which $\overline{\beta_s}$ is the average velocity in the forward direction. For the case of high energy electron, $\overline{\beta_s}$ is infinitely close to 1, i.e. $\overline{\beta_s} \to 1$. Then $z \approx \overline{\beta_s}ct$ and the transverse component of electron trajectory is

$$x(t) \approx -\frac{ec}{\gamma m_e \omega_u^2} \sum_{m=1}^{+\infty} \frac{B_m \sin(m\omega_u t - \delta\phi_m)}{m^2}, \tag{4}$$

in which $\omega_u = \frac{k_u z}{t} \approx k_u c$. The relative transverse velocity is

$$\beta_x = \frac{v_x}{c} = \frac{\dot{x}(t)}{c} \approx -\frac{e}{\gamma m_e \omega_u} \sum_{m=1}^{+\infty} \frac{B_m \cos(m\omega_u t - \delta\phi_m)}{m}. \tag{5}$$

As the energy of the electron is fixed, the electron velocity β is also fixed. Therefore any variation in β_x must result in a corresponding change in β_s because of $\beta^2 = \beta_x^2 + \beta_s^2$. From this we have

$$\overline{\beta_s} \approx 1 - \frac{1}{2\gamma^2} - \frac{e^2}{4\gamma^2 m_e^2 c^2 k_u^2} \sum_{m=1}^{+\infty} \left(\frac{B_m}{m}\right)^2. \tag{6}$$

The fundamental radiation wavelength in the laboratory system [4] is thus

$$\begin{aligned}\lambda_r &= \lambda_u\left(1 - \overline{\beta_s}\cos\vartheta\right)\\ &\approx \frac{\lambda_u}{2\gamma^2}\left\{1 + \frac{e^2}{2m_e^2 c^2 k_u^2} \sum_{m=1}^{+\infty} \left(\frac{B_m}{m}\right)^2 + \gamma^2\vartheta^2\right\}.\end{aligned} \tag{7}$$

In which ϑ is the emission angle respect to the beam direction. Here we define the undulator parameter K as

$$\mathrm{K} = \frac{e}{m_e c k_u} \sqrt{\sum_{m=1}^{+\infty} \left(\frac{B_m}{m}\right)^2}. \tag{8}$$

Then formula (7) returns to the familiar expression as

$$\lambda_r = \frac{\lambda_u}{2\gamma^2}\{1 + \mathrm{K}^2/2 + \gamma^2\vartheta^2\}. \tag{9}$$

Let's consider of the second-order motion solution,

$$\begin{aligned}\ddot{z} &= \frac{e^2 c}{\gamma^2 m_e^2 \omega_u} \sum_{j,k=1}^{+\infty} \frac{B_j B_k \sin(j\omega_u t - \delta\phi_j)\cos(k\omega_u t - \delta\phi_k)}{k}\\ &= \frac{e^2 c}{2\gamma^2 m_e^2 \omega_u} \begin{bmatrix} \sum_{j,k} \frac{B_j B_k \sin(j\omega_u t + k\omega_u t - \delta\phi_j - \delta\phi_k)}{k} \\ + \sum_{j,k} \frac{B_j B_k \sin(j\omega_u t - k\omega_u t - \delta\phi_j + \delta\phi_k)}{k} \end{bmatrix}.\end{aligned} \tag{10}$$

† email address: weitaocaep@qq.com

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So the z component of electron trajectory is written as

$$z = \bar{\beta}_s ct - \frac{e^2 c}{2\gamma^2 m_e^2 \omega_u^3}\left[\sum_{j,k} \frac{B_j B_k \sin(j\omega_u t + k\omega_u t - \delta\phi_j - \delta\phi_k)}{k\,(j+k)^2} + \sum_{j\neq k} \frac{B_j B_k \sin(j\omega_u t - k\omega_u t - \delta\phi_j + \delta\phi_k)}{k\,(j-k)^2}\right]. \quad (11)$$

Combine (4) and (11), the electron motion causes a figure '8' [1] in the co-moving frame.

The spectral angular energy density radiated by an electron [2] in far-field is

$$\frac{d^2W}{d\Omega d\omega} = \frac{e^2\omega^2 N^2}{16\pi^3 c\varepsilon_0} L\left(\frac{N\Delta\omega}{\omega_r}\right)\left|\int_{-\frac{\lambda_u}{2c\bar{\beta}_s}}^{\frac{\lambda_u}{2c\bar{\beta}_s}} \left(\vec{n}\times(\vec{n}\times\vec{\beta})\right) e^{i\omega\left(t-\frac{\vec{n}\cdot\vec{r}}{c}\right)} dt\right|^2 \quad (12)$$

In order to deal with a periodic undulator with N periods, we introduce the distribution function $\left(\frac{N\Delta\omega}{\omega_r}\right) = \frac{\sin^2\left(\frac{N\pi\Delta\omega}{\omega_r}\right)}{N^2\sin^2\left(\frac{\pi\Delta\omega}{\omega_r}\right)}$, in which $\omega_r = \frac{2\pi c}{\lambda_r} \approx \frac{\omega_u}{(1-\bar{\beta}_s\cos\vartheta)}$. For simplification, the energy density on-axis ($\vartheta = 0$) can be expressed as

$$\left.\frac{d^2W}{d\Omega d\omega}\right|_{\substack{\omega=h\omega_r \\ \text{on-axis}}} = \frac{e^2h^2\omega_r^2N^2}{16\pi^3 c\varepsilon_0} L\left(\frac{N\Delta\omega}{\omega_r}\right)\left|\int_{-\frac{\lambda_u}{2c\bar{\beta}_s}}^{\frac{\lambda_u}{2c\bar{\beta}_s}} -\beta_x\, e^{ih\omega_r\left(t-\frac{z}{c}\right)} dt\right|^2$$

$$= \frac{e^4h^2N^2\gamma^2}{4\pi^3\varepsilon_0 c m_e^2(1+K^2/2)^2} L\left(\frac{N\Delta\omega}{\omega_r}\right)\left|\int_{-\frac{\lambda_u}{2c\bar{\beta}_s}}^{\frac{\lambda_u}{2c\bar{\beta}_s}} \sum_m \frac{B_m\cos(m\omega_u t-\delta\phi_m)}{m} e^{i(1-\bar{\beta}_s)h\omega_r t + \frac{ihe^2\omega_r}{2\gamma^2 m_e^2\omega_u^3}\left[\sum_{j,k}\frac{B_jB_k\sin(j\omega_u t+k\omega_u t-\delta\phi_j-\delta\phi_k)}{k\,(j+k)^2} + \sum_{j\neq k}\frac{B_jB_k\sin(j\omega_u t-k\omega_u t-\delta\phi_j+\delta\phi_k)}{k\,(j-k)^2}\right]} dt\right|^2$$

$$= h^2D\left|\int_{-\frac{\lambda_u}{2c\bar{\beta}_s}}^{\frac{\lambda_u}{2c\bar{\beta}_s}} \sum_m \frac{B_m\cos(m\omega_u t-\delta\phi_m)}{m} e^{ih\omega_u t + ihA\left[\sum_{j,k}\frac{B_jB_k\sin(j\omega_u t+k\omega_u t-\delta\phi_j-\delta\phi_k)}{k\,(j+k)^2} + \sum_{j\neq k}\frac{B_jB_k\sin(j\omega_u t-k\omega_u t-\delta\phi_j+\delta\phi_k)}{k\,(j-k)^2}\right]} dt\right|^2 \quad (13)$$

In which

$$D = \frac{e^4N^2\gamma^2}{4\pi^3\varepsilon_0 c m_e^2(1+K^2/2)^2} L\left(\frac{N\Delta\omega}{\omega_r}\right) \text{ and}$$

$$A = \frac{e^2\omega_1}{2\gamma^2 m_e^2\omega_u^3} = \frac{e^2}{m_e^2\omega_u^2(1+K^2/2)} \quad (14)$$

The Bessel function relationship

$$e^{ix\sin\theta} = \sum_{p=-\infty}^{+\infty} J_p(x) e^{ip\theta}, \quad (15)$$

can be used to rewrite (13) as

$$\left.\frac{d^2W}{d\Omega d\omega}\right|_{\substack{\omega=h\omega_r \\ on-axis}} = h^2D\left|\int_{-\frac{\lambda_u}{2c\bar{\beta}_s}}^{\frac{\lambda_u}{2c\bar{\beta}_s}} \sum_m \frac{B_m\cos(m\omega_u t-\delta\phi_m)}{m} e^{ih\omega_u t}\prod_{j,k} e^{\frac{ihAB_jB_k\sin(j\omega_u t+k\omega_u t-\delta\phi_j-\delta\phi_k)}{k\,(j+k)^2}} \prod_{j\neq k} e^{\frac{ihAB_jB_k\sin(j\omega_u t-k\omega_u t-\delta\phi_j+\delta\phi_k)}{k\,(j-k)^2}} dt\right|^2$$

$$= h^2D\left|\int_{-\frac{\lambda_u}{2c\bar{\beta}_s}}^{\frac{\lambda_u}{2c\bar{\beta}_s}} \sum_m \frac{B_m\cos(m\omega_u t-\delta\phi_m)}{m} e^{ih\omega_u t}\prod_{j,k}\left\{\sum_{p_{jka}} J_{p_{jka}}\left[\frac{hAB_jB_k}{k\,(j+k)^2}\right] e^{ip_{jka}(j\omega_u t+k\omega_u t-\delta\phi_j-\delta\phi_k)}\right\} \prod_{j\neq k}\left\{\sum_{p_{jkb}} J_{p_{jkb}}\left[\frac{hAB_jB_k}{k\,(j-k)^2}\right] e^{ip_{jkb}(j\omega_u t-k\omega_u t-\delta\phi_j+\delta\phi_k)}\right\} dt\right|^2 \quad (16)$$

If we set

$$M = \sum_{j,k} p_{jka}(j+k) + \sum_{j\neq k} p_{jkb}(j-k),\ \ \Theta = \sum_{j,k} p_{jka}(\delta\phi_j+\delta\phi_k) + \sum_{j\neq k} p_{jkb}(\delta\phi_j-\delta\phi_k) \quad (17)$$

Then the spectral angular energy density on-axis can be written as

$$\left.\frac{d^2W}{d\Omega d\omega}\right|_{\substack{\omega=h\omega_r \\ on-axis}} = h^2D\left|\sum_m \sum_{h+M=\pm m}\left\{\frac{B_m}{m}\prod_{j,k} J_{p_{jka}}\left(\frac{hAB_jB_k}{k\,(j+k)^2}\right)\prod_{j\neq k} J_{p_{jkb}}\left(\frac{hAB_jB_k}{k\,(j-k)^2}\right)\int_{-\frac{\lambda_u}{2c\bar{\beta}_s}}^{\frac{\lambda_u}{2c\bar{\beta}_s}} \cos(m\omega_u t-\delta\phi_m)\, e^{i(h\omega_u t+M\omega_u t-\Theta)} dt\right\}\right|^2$$

$$= \frac{h^2D\lambda_u^2}{4c^2}\left|\sum_m \sum_{h+M=\pm m}\left\{\frac{B_m}{m}\prod_{j,k} J_{p_{jka}}\left(\frac{hAB_jB_k}{k\,(j+k)^2}\right)\prod_{j\neq k} J_{p_{jkb}}\left(\frac{hAB_jB_k}{k\,(j-k)^2}\right)\cos(-\delta\phi_m \pm \Theta)\right\}\right|^2 \quad (18)$$

COMPARISION WITH SPECTRA SIMULATIONS

For a pure sinusoid magnetic field undulator, $\vec{B} = B_1\sin(k_u z)\,\overrightarrow{e_y}$, the spectral angular energy density on-axis is

$$\left.\frac{d^2W}{d\Omega d\omega}\right|_{\substack{\omega=h\omega_r \\ on-axis}} = \frac{h^2D\lambda_u^2B_1^2}{4c^2\bar{\beta}_s^2}\left|J_{\frac{1-h}{2}}(hAB_1^2/4) + J_{\frac{-1-h}{2}}(hAB_1^2/4)\right|^2. \quad (19)$$

If only the 3rd order magnetic field introduced, $\vec{B} = [B_1\sin(k_u z) + B_3\sin(3k_u z - \delta\phi_3)]\,\overrightarrow{e_y}$, the energy density become more complicated. Considering that high order field strength is much lower than the fundamental one ($B_3 \ll B_1$), the fundamental energy density on-axis can be written as

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$$
\left.\frac{d^2W}{d\Omega d\omega}\right|_{\substack{\omega=\omega_r\\ on-axis}} \approx \frac{h^2 D\lambda_u^2}{4c^2\overline{\beta}_s^2}\cdot
\left|\begin{array}{c}
B_1 J_0\left(\frac{AB_1B_1}{4}\right) J_0\left(\frac{AB_1B_3}{48}\right) J_0\left(\frac{AB_3B_1}{16}\right) J_0\left(\frac{AB_3B_3}{108}\right) J_0\left(\frac{AB_1B_3}{12}\right) J_0\left(\frac{AB_3B_1}{4}\right) + \\
B_1 J_1\left(\frac{AB_1B_1}{4}\right) J_0\left(\frac{AB_1B_3}{48}\right) J_0\left(\frac{AB_3B_1}{16}\right) J_0\left(\frac{AB_3B_3}{108}\right) J_1\left(\frac{AB_1B_3}{12}\right) J_0\left(\frac{AB_3B_1}{4}\right)\cos(\delta\phi_3) + \\
B_1 J_{-1}\left(\frac{AB_1B_1}{4}\right) J_0\left(\frac{AB_1B_3}{48}\right) J_0\left(\frac{AB_3B_1}{16}\right) J_0\left(\frac{AB_3B_3}{108}\right) J_{-1}\left(\frac{AB_1B_3}{12}\right) J_0\left(\frac{AB_3B_1}{4}\right)\cos(\delta\phi_3) + \\
B_1 J_1\left(\frac{AB_1B_1}{4}\right) J_0\left(\frac{AB_1B_3}{48}\right) J_0\left(\frac{AB_3B_1}{16}\right) J_0\left(\frac{AB_3B_3}{108}\right) J_0\left(\frac{AB_1B_3}{12}\right) J_{-1}\left(\frac{AB_3B_1}{4}\right)\cos(\delta\phi_3) + \\
B_1 J_{-1}\left(\frac{AB_1B_1}{4}\right) J_0\left(\frac{AB_1B_3}{48}\right) J_0\left(\frac{AB_3B_1}{16}\right) J_0\left(\frac{AB_3B_3}{108}\right) J_0\left(\frac{AB_1B_3}{12}\right) J_1\left(\frac{AB_3B_1}{4}\right)\cos(\delta\phi_3) + \\
B_1 J_{-1}\left(\frac{AB_1B_1}{4}\right) J_0\left(\frac{AB_1B_3}{48}\right) J_0\left(\frac{AB_3B_1}{16}\right) J_0\left(\frac{AB_3B_3}{108}\right) J_0\left(\frac{AB_1B_3}{12}\right) J_0\left(\frac{AB_3B_1}{4}\right) + \\
B_1 J_0\left(\frac{AB_1B_1}{4}\right) J_0\left(\frac{AB_1B_3}{48}\right) J_0\left(\frac{AB_3B_1}{16}\right) J_0\left(\frac{AB_3B_3}{108}\right) J_1\left(\frac{AB_1B_3}{12}\right) J_0\left(\frac{AB_3B_1}{4}\right)\cos(\delta\phi_3) + \\
B_1 J_0\left(\frac{AB_1B_1}{4}\right) J_0\left(\frac{AB_1B_3}{48}\right) J_0\left(\frac{AB_3B_1}{16}\right) J_0\left(\frac{AB_3B_3}{108}\right) J_0\left(\frac{AB_1B_3}{12}\right) J_{-1}\left(\frac{AB_3B_1}{4}\right)\cos(\delta\phi_3) + \\
\frac{B_3}{3} J_1\left(\frac{AB_1B_1}{4}\right) J_0\left(\frac{AB_1B_3}{48}\right) J_0\left(\frac{AB_3B_1}{16}\right) J_0\left(\frac{AB_3B_3}{108}\right) J_0\left(\frac{AB_1B_3}{12}\right) J_0\left(\frac{AB_3B_1}{4}\right)\cos(\delta\phi_3) + \\
\frac{B_3}{3} J_0\left(\frac{AB_1B_1}{4}\right) J_0\left(\frac{AB_1B_3}{48}\right) J_0\left(\frac{AB_3B_1}{16}\right) J_0\left(\frac{AB_3B_3}{108}\right) J_{-1}\left(\frac{AB_1B_3}{12}\right) J_0\left(\frac{AB_3B_1}{4}\right) + \\
\frac{B_3}{3} J_0\left(\frac{AB_1B_1}{4}\right) J_0\left(\frac{AB_1B_3}{48}\right) J_0\left(\frac{AB_3B_1}{16}\right) J_0\left(\frac{AB_3B_3}{108}\right) J_0\left(\frac{AB_1B_3}{12}\right) J_1\left(\frac{AB_3B_1}{4}\right) + \\
\frac{B_3}{3} J_0\left(\frac{AB_1B_1}{4}\right) J_{-1}\left(\frac{AB_1B_3}{48}\right) J_0\left(\frac{AB_3B_1}{16}\right) J_0\left(\frac{AB_3B_3}{108}\right) J_0\left(\frac{AB_1B_3}{12}\right) J_0\left(\frac{AB_3B_1}{4}\right)\cos(2\delta\phi_3) + \\
\frac{B_3}{3} J_0\left(\frac{AB_1B_1}{4}\right) J_0\left(\frac{AB_1B_3}{48}\right) J_{-1}\left(\frac{AB_3B_1}{16}\right) J_0\left(\frac{AB_3B_3}{108}\right) J_0\left(\frac{AB_1B_3}{12}\right) J_0\left(\frac{AB_3B_1}{4}\right)\cos(2\delta\phi_3)
\end{array}\right|^2 \quad (20)
$$

As shown in Fig.1 is the relative variation of undulator parameter ($\frac{K-K_1}{K}$) due to high-order magnetic field, where K_1 represents the undulator parameter caused by fundamental magnetic field $K_1 = \frac{eB_1}{m_e c k_u}$, and K represents the one described in equation (8). In consideration of the tolerable K error less than 5×10^{-5}, the influence of radiation wavelength caused by high-order magnetic field cannot be ignored.

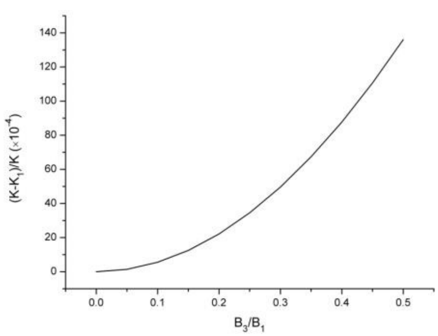

Figure 1: The relative variation of the undulator parameter due to high-order magnetic field. The horizontal coordinate is the ratio of B_3 to B_1.

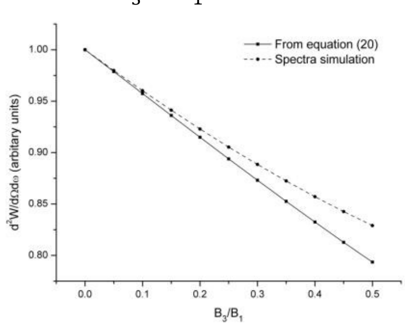

Figure 2: The variation of the fundamental spectral angular energy density with B_3/B_1 (phase $\delta\phi_3 = 0$). The solid line is the one from equation (20), the dashed line comes from Spectra simulation.

In studying the radiation influence of high-order magnetic field, we keep the radiation wavelength unchanged, i.e. the undulator parameter shown in equation (8) remains constant. As shown in Fig.2, if we choose phase shift zero ($\delta\phi_3 = 0$), the fundamental spectral angular energy density on-axis varies with the ratio of B_3 to B_1, and the result is very similar with Spectra [5] simulation.

In fact, the phase cannot be ignored completely. Figure 3 shows the variation of the fundamental spectral angular energy density on-axis with phase. It can be seen the radiation intensity fluctuates according to the magnetic phase.

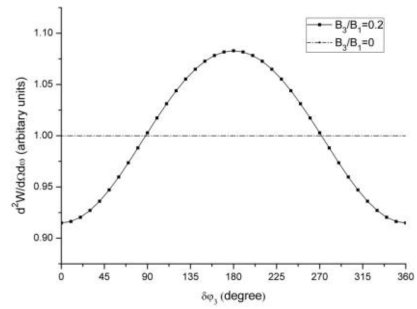

Figure 3: The variation of the fundamental spectral angular energy density with magnetic phase $\delta\phi_3$ ($\frac{B_3}{B1} = 0.2$).

As shown in Fig.3, the magnetic phase will influence the radiation output, the minimum value of radiation intensity appears at $\delta\phi_3 = 0$ and the maximum value at $\delta\phi_3 = \pi$. Both magnetic field distributions are shown in Fig.4, it looks the bigger peak field results in stronger radiation.

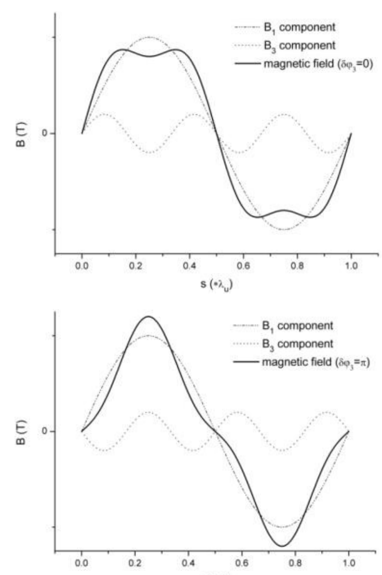

Figure 4: The variation of the fundamental spectral angular energy density with magnetic phase $\delta\phi_3$ ($\frac{B_3}{B1} = 0.2$).

CONCLUSION

The high-order magnetic field is small mount compared with the fundamental one. Normally the ratio is less than 0.2. The influence of high-order magnetic field to the radiation output is lower than 10%.

It is worth noting that the influence of high-order magnetic field to the wavelength cannot be ignored, and the undulator parameter as described in equation (8) is the right one for high-order undualtor.

ISBN: 978-3-95450-199-1 doi:10.18429/JACoW-SAP2017-TUPH09

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